

# “Peeling property” for linearized gravity in null coordinates

Jacek Jezierski\*

Département de Mathématiques, UMR 6083 du CNRS,  
Université de Tours, Parc de Grandmont, F-37200 Tours, France

*on leave of absence:*

Department of Mathematical Methods in Physics,  
University of Warsaw, ul. Hoża 74, 00-682 Warsaw, Poland

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## Abstract

A complete description of the linearized gravitational field on a flat background is given in terms of gauge-independent quasilocal quantities. This is an extension of the results from [14]. Asymptotic spherical quasilocal parameterization of the Weyl field and its relation with Einstein equations is presented. The field equations are equivalent to the wave equation. A generalization for Schwarzschild background is developed and the axial part of gravitational field is fully analyzed. In the case of axial degree of freedom for linearized gravitational field the corresponding generalization of the d’Alembert operator is a Regge-Wheeler equation. Finally, the asymptotics at null infinity is investigated and strong peeling property for axial waves is proved.

## 1 Introduction

We show that seemingly complicated linearized Einstein equations on a Schwarzschild background can be analyzed in terms of gauge invariants. The obtained invariants decouple, in a natural way, into axial and polar parts keeping symmetry with respect to the interchange of the null coordinates  $u$  and  $v$ . The invariant  $\mathbf{y}$  describing axial degrees of freedom, corresponding to  $\Im\Psi_0$ , fulfills Regge-Wheeler equation because axial part of the corresponding component of the Weyl field is gauge-invariant<sup>1</sup>. On the other hand the polar part the Weyl field is not gauge-invariant. However, all components of the Weyl field may be “corrected” in such a way that we obtain invariants which substitute linearized Newman-Penrose scalars.

In [14] we have shown how the gauge-invariant quantities  $\mathbf{x}$ ,  $\mathbf{y}$  describing unconstrained degrees of freedom of the gravitational field arise in a canonical formalism. Here, we concentrate on their relations with linearized Weyl tensor. In the case of a flat background we present, in Theorem 1, an explicit relation between linearized Weyl tensor and the invariants. We continue this analysis for the case of a Schwarzschild background and we show an analogous relation but only for the axial degree of freedom described by  $\mathbf{y}$ . Finally, we examine this result in view of the so-called *peeling* property and we show that axial part of the linearized gravitational field obeys strong peeling.

This paper is organized as follows: In the next Section, some preliminary notions and results for the flat background are introduced. Section 3 contains a generalization for the Schwarzschild background, in particular, gauge invariants and Einstein equations for the axial part of the gravitational field are presented. In Section 4 we discuss the relation between invariants and linearized

\*Partially supported by a grant KBN Nr 2 P03A 047 15 and CNRS Orleans. E-mail: Jacek.Jezierski@fuw.edu.pl

<sup>1</sup>More precisely, some components in axial part of the Weyl field are gauge invariant but not all of them and they have to be “corrected” by metric terms (see Section 4) to become gauge independent.

Riemann tensor. Section 5 is devoted to the investigation of the asymptotics at null infinity for the solutions of Regge-Wheeler equation and its connection with peeling property. To clarify the exposition some of the technical results and proofs have been shifted to the appendix.

## 2 Description in null coordinates for flat background

We present in this section some standard results about linearized gravitational field with nontrivial extensions not only in a notation but also in the framework.

### 2.1 Minkowski metric in null coordinates

Let us consider the flat Minkowski metric of the following form in spherical coordinates

$$\eta_{\mu\nu}dy^\mu dy^\nu = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.1)$$

The Minkowski space  $M$  has a natural structure of a spherical foliation around null infinity, more precisely, the neighbourhood of  $\mathcal{I}^+$  looks like  $S^2 \times M_2$ . We shall use several coordinates on  $M_2$ :  $t, r, \rho, v, u$ . They are defined as follows

$$\rho := r^{-1} \quad u := t - r \quad v := t + r.$$

Let us fix the null coordinates  $(u, v)$  together with the index  $a$  corresponding to them.

The coordinates on a sphere we denote  $(x^A)$ ,  $(A = 1, 2)$ ,  $(x^1 = \theta, x^2 = \phi)$  and the round metric on a unit sphere by  $\overset{\circ}{\gamma}_{AB}$  ( $\overset{\circ}{\gamma}_{AB}dx^A dx^B = d\theta^2 + \sin^2\theta d\phi^2$ ). Let us also denote by  $\overset{\circ}{\Delta}$  the laplacian corresponding to the metric  $\overset{\circ}{\gamma}_{AB}$ . Moreover, we use the symbol “ $\parallel$ ” for the covariant derivative on  $S^2$  with respect to the induced metric  $\eta_{AB}$ .

For convenience we need also some more denotations:  $\rho = r^{-1} = \frac{2}{v-u}$ ,  $\rho_{,a} = \rho^2 \varepsilon_a$  where  $\varepsilon_u := \frac{1}{2}$ ,  $\varepsilon_v := -\frac{1}{2}$ ,  $\eta^{ab}\varepsilon_a \varepsilon_b = 1$ . We define  $\varepsilon^a := \eta^{ab}\varepsilon_b$  and one can check that  $\varepsilon^u = 1$ ,  $\varepsilon^v = -1$ ,  $\eta_{ab}\varepsilon^a \varepsilon^b = 1$ .

The explicit formulae for the components of Minkowski metric can be denoted as follows

$$\eta_{AB} = \rho^{-2} \overset{\circ}{\gamma}_{AB}, \quad \eta_{ab} = -\frac{1}{2}|E_{ab}|, \quad \eta_{aA} = 0$$

where  $E_{uu} = 0 = E_{vv}$  and  $E_{uv} = 1 = -E_{vu}$  and

$$\eta_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}dx^a dx^b + \eta_{AB}dx^A dx^B = -dudv + \rho^{-2}(d\theta^2 + \sin^2\theta d\phi^2).$$

Similarly, the inverse metric has the following components

$$\eta^{AB} = \rho^2 \overset{\circ}{\gamma}^{AB}, \quad \eta^{ab} = -2|E^{ab}|, \quad \eta^{aA} = 0$$

where  $E^{uu} = 0 = E^{vv}$  and  $E^{uv} = 1 = -E^{vu}$ . We shall also need the derivatives

$$\eta^{AB}_{,a} = 2\rho\varepsilon_a \eta^{AB}, \quad \eta_{AB,a} = -2\rho\varepsilon_a \eta_{AB}$$

and finally the nonvanishing Christoffel symbols are the following

$$\Gamma^a_{AB} = \rho\varepsilon^a \eta_{AB}, \quad \Gamma^A_{aB} = -\rho\varepsilon_a \delta^A_B, \quad \Gamma^A_{BC}$$

where  $\Gamma^A_{BC}$  are Christoffel symbols for the spherical covariant derivative “ $\parallel$ ” on  $S^2$ .

## 2.2 Riemann tensor in null coordinates

The linearized Riemann tensor  $R_{\mu\nu\lambda\delta}$  defined in an obvious way in terms of the second derivatives of the linearized metric  $h_{\mu\nu}$  by the formula

$$2R_{\mu\nu\lambda\delta} := h_{\mu\delta;\nu\lambda} - h_{\nu\delta;\mu\lambda} + h_{\nu\lambda;\mu\delta} - h_{\mu\lambda;\nu\delta}$$

has the following components in null coordinates:

$$\begin{aligned} 2R_{abcd} &= h_{ad,bc} - h_{bd,ac} + h_{bc,ad} - h_{ac,bd} \\ 2R_{abcD} &= h_{aD,bc} - h_{bD,ac} + h_{bc,aD} - h_{ac,bD} + \\ &+ \rho\varepsilon_b (h_{aD,c} + h_{cD,a} - h_{ac,D}) - \rho\varepsilon_a (h_{bD,c} + h_{cD,b} - h_{bc,D}) \\ 2R_{AbCd} &= h_{dA||C,b} + h_{bC||A,d} - h_{bd||AC} - h_{AC,bd} + \\ &+ \rho\varepsilon_b (h_{dA||C} - h_{dC||A} - h_{AC,d}) + \rho\varepsilon_d (h_{bC||A} - h_{bA||C} - h_{AC,b}) + \\ &+ \rho\eta_{AC}\varepsilon^a (h_{bd,a} - h_{ad,b} - h_{ab,d}) - 2\rho^2\varepsilon_b\varepsilon_d h_{AC} \\ 2R_{ABCD} &= h_{dA||BC} + h_{BC||A,d} - h_{BD||AC} - h_{AC||B,d} + 2\rho\varepsilon_d (h_{BC||A} - h_{AC||B}) + \\ &+ \rho\eta_{BC}\varepsilon^a (h_{aA,d} - h_{dA,a} + h_{ad,A} + 2\rho\varepsilon_d h_{aA}) - \rho\eta_{AC}\varepsilon^a (h_{aB,d} - h_{dB,a} + h_{ad,B} + 2\rho\varepsilon_d h_{aB}) \\ 2R_{abCD} &= h_{aD||C,b} - h_{bD||C,a} + h_{bC||D,a} - h_{aC||D,b} + \\ &+ 2\rho\varepsilon_b (h_{aD||C} - h_{aC||D}) + 2\rho\varepsilon_a (h_{bC||D} - h_{bD||C}) \\ 2R_{ABCD} &= h_{AD||BC} + h_{BC||AD} - h_{BD||AC} - h_{AC||BD} + \\ &+ \rho\eta_{AC}\varepsilon^a (h_{BD,a} - h_{aB||D} - h_{aD||B}) + \rho\eta_{BD}\varepsilon^a (h_{AC,a} - h_{aC||A} - h_{aA||C}) + \\ &- \rho\eta_{BC}\varepsilon^a (h_{AD,a} - h_{aA||D} - h_{aD||A}) - \rho\eta_{AD}\varepsilon^a (h_{BC,a} - h_{aB||C} - h_{aC||B}) + \\ &+ \rho^2 (h_{BD}\eta_{AC} + h_{AC}\eta_{BD} - h_{AD}\eta_{BC} - h_{BC}\eta_{AD}) + 2\rho^2\varepsilon^a\varepsilon^b h_{ab} (\eta_{AC}\eta_{BD} - \eta_{BC}\eta_{AD}) \end{aligned}$$

We show in the sequel how the above formulae can be generalized for the Schwarzschild background.

## 2.3 Ricci tensor in null coordinates

The linearized Ricci tensor  $R_{\mu\nu} := \eta^{\delta\lambda} R_{\delta\mu\lambda\nu}$  takes the following form in our coordinates

$$\begin{aligned} 2R_{ab} &= h^c_{b,ac} + h^c_{a,cb} - h_{ab}{}^{,c}{}_c - h^c_{c,ab} + h_{aA,b}{}^{||A} + h_{bA,a}{}^{||A} - h_{ab}{}^{||A}{}_A - H_{,ab} + \\ &+ \rho\varepsilon_a H_{,b} + \rho\varepsilon_b H_{,a} + 2\rho\varepsilon^c (h_{ab,c} - h_{ac,b} - h_{bc,a}) \\ 2R_{aB} &= h^b_{B,ab} - h_{aB}{}^{,c}{}_c + h^c_{a,cB} - h^c_{c,aB} + h_a{}^A{}_{||BA} - h_{aB}{}^{||A}{}_A + \chi_B{}^A{}_{||A,a} - \frac{1}{2}H_{||B,a} + \\ &+ \rho\varepsilon_a (2h^b_{B,b} - h^b_{b,B}) - 2\rho\varepsilon^b h_{bB,a} - 2\rho^2\varepsilon_a\varepsilon^b h_{bB} \\ 2R_{AB} &= (h^a_{A||B} + h^a_{B||A})_{,a} - h^a_{a||AB} - \chi_{AB}{}^{,a}{}_a - 2\rho\varepsilon^a \chi_{AB,a} + \chi_A{}^C{}_{||CB} + \chi_B{}^C{}_{||CA} + \\ &- \chi_{AB}{}^{||C}{}_C + \eta_{AB} \left[ -\frac{1}{2}(H^{||C}{}_C + H^{,a}{}_a) + 2\rho\varepsilon^a (H_{,a} - h_a{}^A{}_{||A}) + \rho^2(2\varepsilon^a\varepsilon^b h_{ab} - H) \right] \end{aligned}$$

where  $H := \eta^{AB} h_{AB}$  and  $\chi_{AB} := h_{AB} - \frac{1}{2}\eta_{AB}H$ . Some components of the linearized Ricci are also derived for Schwarzschild background in Appendix D.

## 2.4 Gauge in null coordinates

The gauge transformation  $\xi_\mu$

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + 2\xi_{(\mu;\nu)}$$

splits in the following way

$$\begin{aligned} h_{ab} &\longrightarrow h_{ab} + \xi_{a,b} + \xi_{b,a} \\ h_{aA} &\longrightarrow h_{aA} + \xi_{a,A} + \xi_{A,a} + 2\rho\varepsilon_a\xi_A \\ h_{AB} &\longrightarrow h_{AB} + \xi_{A||B} + \xi_{B||A} - 2\rho\eta_{AB}\varepsilon^a\xi_a. \end{aligned} \tag{2.2}$$

There are also some useful formulae

$$\begin{aligned} \chi_{AB} &\longrightarrow \chi_{AB} + \xi_{A||B} + \xi_{B||A} - \eta_{AB}\xi^C{}_{||C} \\ \frac{1}{2}H &\longrightarrow \frac{1}{2}H + \xi^A{}_{||A} - 2\rho\varepsilon^a\xi_a \\ h_a{}^A &\longrightarrow h_a{}^A + \xi_a{}^{||A} + \xi^A{}_{,a} \end{aligned}$$

which are straightforward consequences of the previous ones.

## 2.5 Invariants and vacuum Einstein equations

Let us introduce the following gauge invariant quantities<sup>2</sup>

$$\mathbf{y}_a := (\overset{\circ}{\Delta} + 2)h_{aA||B}\varepsilon^{AB} - (\rho^{-2}\chi_A{}^C{}_{||CB}\varepsilon^{AB})_{,a} \tag{2.3}$$

$$\mathbf{y} := 2\rho^{-2}(h_{bB||A}\varepsilon^{AB})_{,a}E^{ab} \tag{2.4}$$

$$\mathbf{x} := \rho^{-2}\chi^{AB}{}_{||BA} - \frac{1}{2}\overset{\circ}{\Delta}H + \rho^{-1}\varepsilon^a H_{,a} - H + 2\varepsilon^a\varepsilon^b h_{ab} - 2\rho^{-1}\varepsilon^a h_a{}^A{}_{||A} \tag{2.5}$$

$$\begin{aligned} \mathbf{x}_{ab} &:= \overset{\circ}{\Delta}(\overset{\circ}{\Delta} + 2)h_{ab} - (\overset{\circ}{\Delta} + 2)[(\rho^{-2}h_a{}^A{}_{||A})_{,b} + (\rho^{-2}h_b{}^A{}_{||A})_{,a}] \\ &\quad + [\rho^{-2}(\rho^{-2}\chi^{AB}{}_{||AB})_{,a}]_{,b} + [\rho^{-2}(\rho^{-2}\chi^{AB}{}_{||AB})_{,b}]_{,a} \end{aligned} \tag{2.6}$$

where  $\varepsilon^{AB}$  is the Levi-Civita skew-symmetric tensor on a sphere  $\{u = \text{const.}, v = \text{const.}\}$  such that  $\rho^{-2}\sin\theta\varepsilon^{12} = 1$ .

The axial invariants are not independent they are related as follows:

$$2\rho^{-2}\mathbf{y}_{b,a}E^{ab} = (\overset{\circ}{\Delta} + 2)\mathbf{y}$$

which is a simple consequence of the definition (2.4).

If we assume that vacuum Einstein equations  $R_{\mu\nu} = 0$  are fulfilled, we obtain the following equations for our invariants:

The axial part reads as

$$(\rho^{-2}\mathbf{y}^a)_{,a} = 2\rho^{-4}\overset{\circ}{R}_A{}^B{}_{||BD}\varepsilon^{AD} = 0 \tag{2.7}$$

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<sup>2</sup>We leave to the reader an exercise to check that those quantities are gauge invariant (using (2.2)), however, in the Appendix B we show this property in a more general case.

$$2E^{ab}(\rho^{-2}\mathbf{y})_{,b} + \rho^{-2}\mathbf{y}^a = -2\rho^{-4}R^a{}_{B||D}\varepsilon^{BD} = 0 \quad (2.8)$$

or takes another form in terms of the quantity  $\mathbf{y}^a$

$$[\rho^{-4}(\mathbf{y}_{a,b} - \mathbf{y}_{b,a})]^{,b} + \rho^{-2}(\overset{\circ}{\Delta} + 2)\mathbf{y}_a = -2\rho^{-4}(\overset{\circ}{\Delta} + 2)R_{aB||D}\varepsilon^{BD} = 0.$$

The polar part takes the following form

$$\begin{aligned} \mathbf{x}^{ab}{}_{,ab} - \rho^2 \overset{\circ}{\Delta}(\overset{\circ}{\Delta} + 2)\mathbf{x} &= 4(\rho^{-4} \overset{\circ}{R}{}^{AB}{}_{||BA})^{,a}{}_a + \overset{\circ}{\Delta}(\overset{\circ}{\Delta} + 2)(\eta^{ab}R_{ab} - \eta^{AB}R_{AB}) = 0 \\ \eta^{ab}\mathbf{x}_{ab} &= 4\rho^{-4} \overset{\circ}{R}{}^{AB}{}_{||BA} = 0 \end{aligned} \quad (2.9)$$

$$\mathbf{x}_{ab} - 2(\rho^{-2}\mathbf{x})_{,ab} + \eta_{ab}(\rho^{-2}\mathbf{x})^{,c}{}_c = 0. \quad (2.10)$$

The left-hand side of the last equation (2.10) depends on all seven polar components of the Ricci tensor  $R_{ab}$ ,  $R^{AB}{}_{||B}$ ,  $\eta^{AB}R_{AB}$ ,  $\overset{\circ}{R}{}^{AB}{}_{||BA}$ . Assuming that all of them are vanishing one can show that (2.10) is true. From eq. (2.10) and (2.8) we conclude that the invariants  $\mathbf{x}_{ab}$  and  $\mathbf{y}_a$  depend locally on  $\mathbf{x}$ ,  $\mathbf{y}$ . More precisely,

$$\begin{aligned} \mathbf{x}_{ab} &= 2(\rho^{-2}\mathbf{x})_{,ab} - \eta_{ab}(\rho^{-2}\mathbf{x})^{,c}{}_c \\ \mathbf{y}^a &= -2E^{ab}(\rho^{-2}\mathbf{y})_{,b}\rho^2 \end{aligned}$$

and the primary data  $(\mathbf{x}, \mathbf{y})$  fulfills usual wave equation.

$$(\rho^{-1}\mathbf{y})^{,a}{}_a + \rho \overset{\circ}{\Delta}\mathbf{y} = 0 \quad (\rho^{-1}\mathbf{x})^{,a}{}_a + \rho \overset{\circ}{\Delta}\mathbf{x} = 0$$

We describe in the sequel how the full Riemann tensor can be reconstructed from the invariants  $\mathbf{x}, \mathbf{y}$ .

## 2.6 Quasi-local relations between gauge invariants and linearized Riemann or Weyl tensor

It is convenient to use skew-symmetric tensor  $\varepsilon^{ab}$  instead of density  $E^{ab}$ . It can be defined as follows

$$\sqrt{|\det \eta_{ab}|}\varepsilon^{uv} = 1; \quad \frac{\varepsilon_{uv}}{\sqrt{|\det \eta_{ab}|}} = -1 \quad \varepsilon^{ab} = 2E^{ab}$$

One can show that the linearized Riemann tensor has the following  $(2+2)$  ‘‘spherical decomposition’’. In terms of our invariants it decouples into axial part

$$\begin{aligned} \frac{1}{2}\rho^{-2}\varepsilon^{ab}\varepsilon^{CD}R_{abCD} &= \mathbf{y} \\ R_{abcD||E}\varepsilon^{ab}\varepsilon^{DE} &= \rho^3(\rho^{-1}\mathbf{y})_{,c} \\ \varepsilon^{AB}R_{AB}{}^C{}_{d||C} &= \varepsilon_d{}^b\rho^3(\rho^{-1}\mathbf{y})_{,b} \\ 4\rho^{-4} \overset{\circ}{R}{}^A{}_{bCd||AD}\varepsilon^{CD} &= (\rho^{-2}\mathbf{y}_d)_{,b} + (\rho^{-2}\mathbf{y}_b)_{,d} \end{aligned} \quad (2.11)$$

and polar part

$$\begin{aligned} \rho^{-2}\eta^{AC}\eta^{BD}R_{ABCD} &= \frac{1}{2}\rho^{-2}\varepsilon^{AB}\varepsilon^{CD}R_{ABCD} = \mathbf{x} \\ -2\rho^{-2}\varepsilon^{ab}\varepsilon^{cd}R_{abcd} &= \rho^{-2}\eta^{ac}\eta^{bd}R_{abcd} = -\rho^{-2}\eta^{ac}\eta^{BD}R_{aBcD} = \mathbf{x} \end{aligned}$$

$$\begin{aligned}
\rho^{-2}\varepsilon^{AB}R_{ABCD}||_E\varepsilon^{CE} &= \mathbf{x}_{,d} \\
4\rho^{-4}\overset{\circ}{R}{}^A{}_b{}^C{}_d||_{AC} &= -\mathbf{x}_{bd} \\
\rho^{-2}(\overset{\circ}{\Delta} + 2)R_{abc}{}^D||_D\varepsilon^{ab} &= -\rho^{-1}(\rho\mathbf{x}_{ac})_{,b}\varepsilon^{ab} \\
2\rho^{-2}(\overset{\circ}{\Delta} + 2)R^A{}_{bAd} &= \mathbf{x}_{bd} + (\rho^{-2}\mathbf{x}_{,d})_{,b} + (\rho^{-2}\mathbf{x}_{,b})_{,d}
\end{aligned} \tag{2.12}$$

In above formulae, as in the whole paper, we use extensively some operators on a unit sphere which become isomorphisms when we assume that mono-dipole part of the field vanishes (see here Appendix E and also [2] or Appendix B in [5]). The above equations contain the full information on ten independent components of the Weyl tensor up to the mono-dipole part of the field<sup>3</sup>. Moreover, one can easily check the “peeling” property [19] at  $\mathcal{I}^+$  starting from the invariants  $\mathbf{x}, \mathbf{y}$  as a primary data. More precisely, assuming the following expansion

$$\mathbf{x} = \mathbf{x}_1\rho + \mathbf{x}_2\rho^2 + \mathbf{x}_3\rho^3 + \mathbf{x}_4\rho^4 + \dots \tag{2.13}$$

and the same form for  $\mathbf{y}$

$$\mathbf{y} = \mathbf{y}_1\rho + \mathbf{y}_2\rho^2 + \mathbf{y}_3\rho^3 + \mathbf{y}_4\rho^4 + \dots \tag{2.14}$$

we have

$$\begin{aligned}
\mathbf{x}_u &= -\dot{\mathbf{x}}_1\rho + \left(\frac{1}{2}\mathbf{x}_1 - \dot{\mathbf{x}}_2\right)\rho^2 + \dots \\
\mathbf{x}_v &= \frac{1}{2}\mathbf{x}_1\rho^2 - \frac{1}{2}\mathbf{x}_3\rho^4 - \mathbf{x}_4\rho^5 + \dots \\
\mathbf{x}_{uu} &= 2\ddot{\mathbf{x}}_1\rho^{-1} + 2(\ddot{\mathbf{x}}_2 - \dot{\mathbf{x}}_1) + \dots \\
\mathbf{x}_{vv} &= \frac{1}{2}\mathbf{x}_3\rho^3 + 3\mathbf{x}_4\rho^4 + \dots
\end{aligned}$$

We summarize below in the table the relation of our invariants with the Newman-Penrose [18] scalars and the Christodoulou-Klainerman-Nicolò [15] decomposition of the Weyl tensor:

Price	Weyl	C – K – N	polar	axial	N – P	asymptotics
$\Psi_2$	$\overset{\circ}{W}{}_v{}^A{}_v{}^B$	$\rho^{-2}\alpha_{\text{CKN}}$	$\rho^2\mathbf{x}_{vv}$	$\rho^2(\rho^{-2}\mathbf{y}_v)_{,v}$	$\Psi_0$	$\rho^5(\mathbf{x}_3, \mathbf{y}_3)$
$\Psi_{-2}$	$\overset{\circ}{W}{}_u{}^A{}_u{}^B$	$\rho^{-2}\underline{\alpha}_{\text{CKN}}$	$\rho^2\mathbf{x}_{uu}$	$\rho^2(\rho^{-2}\mathbf{y}_u)_{,u}$	$\Psi_4$	$\rho(\ddot{\mathbf{x}}_1, \ddot{\mathbf{y}}_1)$
$\Psi_1$	$\rho^{-1}\varepsilon^{ab}W_{abv}{}^A$	$\rho^{-1}\beta_{\text{CKN}}$	$\rho^2(\rho^{-1}\mathbf{x})_{,v}$	$\rho^2(\rho^{-1}\mathbf{y})_{,v}$	$\Psi_1$	$\rho^4(\mathbf{x}_2, \mathbf{y}_2)$
$\Psi_{-1}$	$\rho^{-1}\varepsilon^{ab}W_{abu}{}^A$	$\rho^{-1}\underline{\beta}_{\text{CKN}}$	$\rho^2(\rho^{-1}\mathbf{x})_{,u}$	$\rho^2(\rho^{-1}\mathbf{y})_{,u}$	$\Psi_3$	$\rho^2(\dot{\mathbf{x}}_1, \dot{\mathbf{y}}_1)$
$\Psi_0$	$W^a{}_{bcd}, W^A{}_{Bcd}$	$\rho_{\text{CKN}}, \sigma_{\text{CKN}}$	$\rho^2\mathbf{x}$	$\rho^2\mathbf{y}$	$\Psi_2$	$\rho^3(\mathbf{x}_1, \mathbf{y}_1)$

where  $\overset{\circ}{W}{}^A{}_{abB} = W^A{}_{abB} - \frac{1}{2}\delta^A{}_B W^D{}_{abD}$  (cf.  $TS$  transformation in appendix) and  $W^\nu{}_{\mu\lambda\delta}$  is the linearized Weyl tensor.

One can easily check that among all  $\Psi$ ’s only  $\rho^{-2}\Psi_0^{\text{Price}} = \rho^{-2}\Psi_2^{\text{NP}} = \mathbf{x} + i\mathbf{y}$  fullfills usual wave equation. The mono-dipole-free parts of the invariants  $\mathbf{x}, \mathbf{y}$  correspond to the unconstrained degrees of freedom of the linearized gravitational field. Moreover, they play a natural role of the “positions” in the reduced initial data set on a Cauchy surface ([11], [13], [14]).

**Remark** The Teukolsky equations [22] for  $\Psi_0, \Psi_4$  on a Kerr background seem to be quite strange as primary equations because they are not deformations of the usual wave equation when we pass to the asymptotically flat region. We would like to stress that there exists a generalization for the notion of  $\mathbf{x}$  and  $\mathbf{y}$  on a Schwarzschild background and both invariants fulfill a deformed wave equation – Regge-Wheeler for  $\mathbf{y}$  and Zerilli for  $\mathbf{x}$  (see [14]).

<sup>3</sup> The mono-dipole part is discussed in [12] and it corresponds to the charges related to the Poincaré group.

**Theorem 1** *The linearized Riemann tensor for the vacuum Einstein equations depends quasilocally on the invariants  $(\mathbf{x}, \mathbf{y})$  which contain the full information about the linearized gravitational field. Moreover, the invariants  $\mathbf{x}$  and  $\mathbf{y}$  fulfill usual wave equation.*

In other words any mono-dipole-free solution  $(\mathbf{x}, \mathbf{y})$  of the wave equation gives a Weyl field:

$$\begin{aligned}
W_{abcd} &= -\frac{1}{2}\rho^2 \mathbf{x} \varepsilon_{ab} \varepsilon_{cd} \\
W_{ABcd} &= -\frac{1}{2}\rho^2 \mathbf{y} \varepsilon_{AB} \varepsilon_{cd} \\
W_a{}^B{}_{cd||B} &= -\frac{1}{2}\varepsilon_{cd} \varepsilon^b{}_a \rho^3 (\rho^{-1} \mathbf{x})_{,b} \\
W_{aBcd||E} \varepsilon^{BE} &= -\frac{1}{2}\varepsilon_{cd} \rho^3 (\rho^{-1} \mathbf{y})_{,a} \\
\overset{\circ}{W}_c{}^{AB}{}_{d||AB} &= \frac{1}{4}\rho^4 \mathbf{x}_{cd} = \frac{1}{2}\rho^4 \left[ (\rho^{-2} \mathbf{x})_{,cd} - \frac{1}{2}\eta_{cd} (\rho^{-2} \mathbf{x})^{,b}{}_{,b} \right] \\
\overset{\circ}{W}_c{}^A{}_{Bd||AC} \varepsilon^{BC} &= \frac{1}{4}\rho^4 [\varepsilon_c{}^b (\rho^{-2} \mathbf{y})_{,bd} + \varepsilon_d{}^b (\rho^{-2} \mathbf{y})_{,bc}]
\end{aligned} \tag{2.15}$$

and  $W^\nu{}_{\mu\lambda\delta}$  fulfils field equations given by Bianchi identities. The formulae (2.15) are also valid (outside origin) if we include mono-dipole part of the fields  $\mathbf{x}$  and  $\mathbf{y}$  (see [12]). More precisely, a mono-dipole solution  $\mathbf{x} = 4\mathbf{m}\rho + 12\mathbf{k}\rho^2$ ,  $\mathbf{y} = 12\mathbf{s}\rho^2$  corresponds to the 10 Poincaré charges (a monopole and three dipoles):  $\mathbf{m}$  — mass,  $\mathbf{s}$  — spin,  $\mathbf{k}$  — center of mass and  $\mathbf{p}$  — linear momentum which is related with center of mass by the relation  $\mathbf{p} = (\partial_u + \partial_v)\mathbf{k}$ . Moreover, the “charges” fulfill the following equations:  $\partial_a \mathbf{m} = \partial_a \mathbf{s} = \partial_a \mathbf{p} = 0$ ,  $\overset{\circ}{\Delta} \mathbf{m} = (\overset{\circ}{\Delta} + 2)\mathbf{p} = (\overset{\circ}{\Delta} + 2)\mathbf{s} = (\overset{\circ}{\Delta} + 2)\mathbf{k} = 0$  which simply mean that  $\mathbf{m}$  is a constant and  $\mathbf{p}$ ,  $\mathbf{s}$  are constant dipoles.

**Remark** One can check that the “exterior” bounds (10.19 -10.20) for the various null components of the Weyl field in [15] (p. R114-R115) are related to the most important radiative terms  $\mathbf{x}_1$  and  $\mathbf{y}_1$  which appear in  $\underline{\alpha}_{\text{CKN}}$ ,  $\underline{\beta}_{\text{CKN}}$ ,  $\rho_{\text{CKN}}$  and  $\sigma_{\text{CKN}}$  but the fall-off conditions on  $\alpha_{\text{CKN}}$  and  $\beta_{\text{CKN}}$  are slightly weaker. In particular it is not known how to improve the fall-off for  $\alpha_{\text{CKN}}$  in the case of curved space-time. This leads to the problem in the standard conformal approach where stronger fall-off on  $\alpha_{\text{CKN}}$ <sup>4</sup> is assumed.

## 2.7 Hierarchy of asymptotic solution on scri for scalar wave equation

Let us consider the wave equation in null coordinates  $(u, v)$

$$\rho^{-1}(\rho^{-1}\varphi)^{,a}{}_a + \overset{\circ}{\Delta}\varphi = 0 \tag{2.16}$$

and suppose we are looking for a solution of the wave equation (2.16) as a series (see [8])

$$\varphi = \varphi_1 \rho + \varphi_2 \rho^2 + \varphi_3 \rho^3 + \dots \tag{2.17}$$

where each  $\varphi_n$  is a function on  $\mathcal{I}^+$ ,  $\partial_v \varphi_n = 0$ .

If we put the series (2.17) into the wave equation (2.16), we obtain the following recursion

$$\partial_u \varphi_{n+1} = -\frac{1}{2n} [\overset{\circ}{\Delta} + (n-1)n] \varphi_n. \tag{2.18}$$

The above formula is the same as equations 2, 3, 4 in [3] and equation 1.6 in [8] but written in a more elegant way. Let us finish this Section with the following Remark which is devoted to

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<sup>4</sup>corresponding to  $\Psi_0^{\text{NP}}$

so-called *Newman-Penrose constants*:

**Remark.** The kernel of the operator  $[\overset{\circ}{\Delta} + l(l+1)]$  corresponds to the  $l$ -th spherical harmonics. The right-hand side of (2.18) vanishes on the  $n-1$  spherical harmonics subspace. This means that the corresponding multipole in  $\varphi_{n+1}$  does not depend on  $u$ . In particular, for  $n=3$  we have quadrupole charge in the fourth order. The nonlinear counterpart of this object is called *Newman-Penrose constant* (see [18], [4], [13]). In particular, in our case, this constant is related to the quadrupole part of  $\mathbf{x}_4$  and  $\mathbf{y}_4$ .

### 3 Gravitational field on a Schwarzschild background

In [14] it is shown how to generalize the gauge-invariant quantities  $\mathbf{x}$  and  $\mathbf{y}$  for the case of Schwarzschild background. We would like to investigate the following problems:

1. What is the relation (similar to (2.15)) between linearized Weyl tensor and the gauge invariants? and
2. How this relation applied to the asymptotics for the solutions of deformed wave equation (Regge-Wheeler and Zerilli) clarifies a peeling property for the linearized Weyl tensor?

In this section we summarize results of Adam Jankowski [10] who analyzed axial degree of freedom. We shall often use the same letters for the generalizations of the objects from flat to Schwarzschild background with the obvious identification when the mass  $m=0$ .

#### 3.1 Schwarzschild metric in null coordinates

Let us start with spherical coordinates  $r, t, \theta, \phi$ , hence the Schwarzschild metric has the following form:

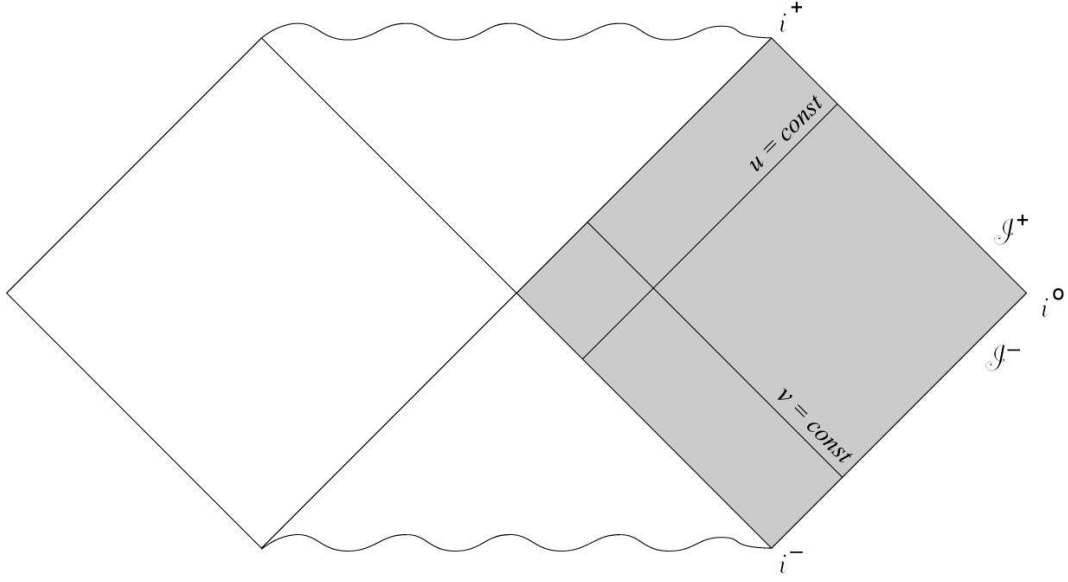
$$\eta_{\mu\nu} dx^\mu dx^\nu = - \left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (3.1)$$

Spacetime  $M$  has a natural foliation with respect to spherical symmetry, more precisely, it splits into  $S^2 \times M_2$ , where  $S^2$  is a two-dimensional sphere and  $M_2$  is described by coordinates  $r$  and  $t$ . This way spacetime  $(M, \eta)$  with coordinates  $(u, v, \theta, \phi)$  splits into  $(S^2, \eta|_{S^2})$  and  $(M_2, \eta|_{M_2})$ . we shall often denote by  $x^A$  coordinates  $(\theta, \phi)$  on a sphere, but on  $M_2$  we use null coordinates defined in terms of standard  $r$  and  $t$  as follows:

$$\begin{aligned} u &= t - r - 2m \ln(r - 2m) & u \in ] - \infty, +\infty[ \\ v &= t + r + 2m \ln(r - 2m) & v \in ] - \infty, +\infty[ \end{aligned} \quad (3.2)$$

Radial curves ( $\theta = \text{const.}, \phi = \text{const.}$ ) with fixed  $u = \text{const.}$  are null geodesics, similarly for  $v = \text{const.}$





Our main interest will concentrate on the domain  $r > 2m$  marked by grey colour on the above *Carter–Penrose diagram for Schwarzschild spacetime*.

Let us denote coordinates  $(u, v)$  by small Latin characters  $(x^a)$ , spherical coordinates by capital Latin characters  $(x^A)$ ,  $(A = 1, 2)$ ,  $(x^1 = \theta, x^2 = \phi)$ , and the unit round spherical metric by  $\overset{\circ}{\gamma}_{AB}$ :

$$\overset{\circ}{\gamma}_{AB} dx^A dx^B = d\theta^2 + \sin^2 \theta d\phi^2.$$

Let us also notice that  $M_2 \perp S^2$ , because  $\eta_{aA} = 0$ .

Let us fix some more notation: symbol “ $\overset{\circ}{|}$ ” denotes two-dimensional covariant derivative on  $S^2$  compatible with induced metric  $\eta_{AB}$ , the equality

$$\eta_{AB} = r^2 \overset{\circ}{\gamma}_{AB}$$

implies that the Christoffel symbols for  $\eta_{AB}$  and  $\overset{\circ}{\gamma}_{AB}$  are the same. Moreover, symbol  $\overset{\circ}{\Delta}$  denotes Beltrami-Laplace operator for the unit metric  $\overset{\circ}{\gamma}_{AB}$ . We keep the same notation as in Section 2 with obvious generalization from flat background to Schwarzschild one.

$$\eta_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} dx^a dx^b + \eta_{AB} dx^A dx^B \quad (3.3)$$

$$\eta_{\mu\nu} dx^\mu dx^\nu = - \left( 1 - \frac{2m}{r} \right) dudv + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (3.4)$$

From definition (3.2) we have  $\frac{1}{2}(dv - du) = k^{-1}dr$  where by  $k$  we denote:

$$k := 1 - \frac{2m}{r}.$$

We shall analyze the region far away from the sources and assume that  $r > 2m$  which implies  $k > 0$ . Using (3.2) one can verify that

$$r_{,a} = -\varepsilon_a \sqrt{k}, \quad (3.5)$$

where  $\varepsilon_a$  is defined as follows:

$$\varepsilon_v = -\frac{1}{2}k^{\frac{1}{2}} \quad \varepsilon_u = \frac{1}{2}k^{\frac{1}{2}} \quad (3.6)$$

$$\varepsilon^v = -k^{-\frac{1}{2}} \quad \varepsilon^u = k^{-\frac{1}{2}} \quad (3.7)$$

and the indices are raised by the two-dimensional inverse metric  $\eta^{ab}$ . This way we have introduced a unit-length radial vector  $\varepsilon^a$

$$\eta_{ab}\varepsilon^a\varepsilon^b = \eta^{ab}\varepsilon_a\varepsilon_b = 1.$$

One can also check that

$$\eta_{AB,a} = -\frac{2\sqrt{k}}{r}\varepsilon_a\eta_{AB} \quad (3.8)$$

and we have the following nonvanishing Christoffel symbols for the metric connection on  $M$ :

$$\Gamma^A_{Ba} = -\delta_B^A \frac{\sqrt{k}}{r}\varepsilon_a, \quad (3.9)$$

$$\Gamma^a_{AB} = \eta_{AB} \frac{\sqrt{k}}{r}\varepsilon^a, \quad (3.10)$$

$$\Gamma^A_{BC} = \frac{1}{2}\eta^{AD}(\eta_{DB,C} + \eta_{DC,B} - \eta_{CB,D}), \quad (3.11)$$

$$\Gamma^a_{bc} = \frac{m}{r^2}k^{\frac{1}{2}}(-\delta^a_b\varepsilon_c - \delta^a_c\varepsilon_b + \varepsilon^a\eta_{bc}). \quad (3.12)$$

Let us notice that  $\Gamma^A_{BC}$  are simultaneously Christoffel symbols for the induced two-dimensional metric  $\eta|_{S^2}$ . More precisely, in usual angular coordinates  $(\theta, \phi)$  we have two nonvanishing components:  $\Gamma^\phi_{\phi\theta} = \cot\theta$  and  $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ .

We can assign the symbols  $\Gamma^A_{BC}$  and  $\Gamma^a_{bc}$  to covariant derivative on  $S^2$  and  $M_2$  respectively, and denote (on the example of a covector) as follows

$$\xi_{a\ddagger b} := \partial_b\xi_a - \xi_f\Gamma^f_{ab} \quad (3.13)$$

$$\xi_{A||B} := \partial_B\xi_A - \xi_F\Gamma^F_{AB} \quad (3.14)$$

Riemann tensor for the metric  $\eta_{\mu\nu}$  equals to Weyl (Schwarzschild metric is a vacuum solution) and has the following nonvanishing components (up to the symmetries):

$$C^A_{BCD} = \frac{2m}{r^3}(\delta_C^A\eta_{BD} - \delta_D^A\eta_{BC}), \quad (3.15)$$

$$C^a_{bcd} = \frac{2m}{r^3}(\delta_c^a\eta_{bd} - \delta_d^a\eta_{bc}), \quad (3.16)$$

$$C^A_{aBb} = -\frac{m}{r^3}\delta_B^A\eta_{ab}. \quad (3.17)$$

Levi-Civita connection on  $S^2$  has the following components of the two-dimensional Riemann tensor:

$${}^2R^A_{BCD} = \frac{1}{r^2}(\delta_C^A\eta_{BD} - \delta_D^A\eta_{BC}). \quad (3.18)$$

Similarly, the corresponding curvature for the metric  $\eta_{ab}$  has the form:

$${}^2R^a_{bcd} = -\frac{2m}{r^3}(\delta_c^a\eta_{bd} - \delta_d^a\eta_{bc}). \quad (3.19)$$

Let us introduce Levi-Civita tensor for  $(S^2, \eta|_{S^2})$  and  $(M_2, \eta|_{M_2})$  respectively:

$$\varepsilon^{AB} := \frac{E^{AB}}{\sqrt{|\det \eta_{CD}|}} \quad \varepsilon^{ab} := \frac{E^{ab}}{\sqrt{|\det \eta_{cd}|}}$$

where  $E^{ab}$  are coordinates of tensor density  $\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}$  so the values are in set  $\{-1, 0, 1\}$ . Similarly,  $E^{AB}$  are coordinates for  $\frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial \phi}$ , so we may write explicitly

$$\varepsilon^{AC} = \frac{E^{AB}}{r^2 \sin \theta}, \quad (3.20)$$

$$\varepsilon^{ab} = \frac{2E^{ab}}{k}. \quad (3.21)$$

Let us notice that the metric  $\eta_{ab}$  has a signature  $(+, -)$ , hence its determinant is negative:

$$\det \eta_{ab} < 0.$$

This is important when we raise indices in  $\varepsilon_{ab}$ :

$$\varepsilon_{ab} = \eta_{ac} \eta_{bd} \varepsilon^{cd} = -\sqrt{|\det \eta_{ab}|} E_{ab} = -\frac{1}{2} k E_{ab}, \quad (3.22)$$

but on  $S^2$  the corresponding sign is positive:

$$\varepsilon_{AB} = \eta_{AC} \eta_{BD} \varepsilon^{CD} = \sqrt{|\det \eta_{AB}|} E_{AB} = r^2 \sin \theta E_{AB}. \quad (3.23)$$

It would be useful to derive explicit formulae for first and second derivatives of  $\varepsilon^{AC}$  and  $\eta^{AC}$  with respect to null coordinates on  $M_2$  which are implied by (3.5):

$$\varepsilon^{AC}{}_{\dagger a} = \frac{2\sqrt{k}}{r} \varepsilon^{AC} \varepsilon_a, \quad \eta^{AC}{}_{\dagger a} = \frac{2\sqrt{k}}{r} \eta^{AC} \varepsilon_a \quad (3.24)$$

$$\varepsilon^{AC}{}_{\dagger ab} = \left( \frac{6k}{r^2} \varepsilon_a \varepsilon_b - \frac{2m}{r^3} \eta_{ab} \right) \varepsilon^{AC}, \quad \eta^{AC}{}_{\dagger ab} = \left( \frac{6k}{r^2} \varepsilon_a \varepsilon_b - \frac{2m}{r^3} \eta_{ab} \right) \eta^{AC}, \quad (3.25)$$

where obviously the objects  $\varepsilon^{AC}$  and  $\eta^{AC}$  are scalars with respect to the covariant derivative (3.14).

### 3.2 Gauge transformation for the linearized metric tensor $h_{\mu\nu}$

We shall analyze, from  $2+2$  decomposition point of view, the gauge transformation generated by infinitesimal diffeomorphism of  $M$  for the linearized metric tensor<sup>5</sup>:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\xi_{(\mu;\nu)}. \quad (3.26)$$

Let us first split the covariant derivatives of the covector field  $\xi_\mu = (\xi_a, \xi_A)$ :

$$\xi_{a;b} = \partial_b \xi_a - \xi_f \Gamma^f{}_{ab} = \xi_{a\dagger b}$$

$$\xi_{a;A} = \partial_A \xi_a - \xi_F \Gamma^F{}_{aA} = \xi_{a||A} + \frac{\sqrt{k}}{r} \xi_A \varepsilon_a$$

$$\xi_{A;b} = \partial_b \xi_A - \xi_F \Gamma^F{}_{bA} = \xi_{A\dagger b} + \frac{\sqrt{k}}{r} \xi_A \varepsilon_b$$

$$\xi_{A;B} = \partial_B \xi_A - \xi_F \Gamma^F{}_{AB} - \xi_f \Gamma^f{}_{AB} = \xi_{A||B} - \frac{\sqrt{k}}{r} \xi_f \varepsilon^f$$

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<sup>5</sup>The standard linearization formulae have been shifted to the Appendix A.

and next apply them to gauge transformation of the tensor  $h_{\mu\nu}$ :

$$h_{ab} \rightarrow h_{ab} + \xi_{a\ddagger b} + \xi_{b\ddagger a}, \quad (3.27)$$

$$h_{aA} \rightarrow h_{aA} + \xi_{a||A} + \xi_{A\ddagger a} + \frac{2\sqrt{k}}{r}\varepsilon_a\xi_A, \quad (3.28)$$

$$h_{AB} \rightarrow h_{AB} + \xi_{A||B} + \xi_{B||A} - \frac{2\sqrt{k}}{r}\xi_c\varepsilon^c\eta_{AB}. \quad (3.29)$$

Here the symbol  $\circ$  over two-dimensional tensor  $t_{AB}$  denotes its traceless part:

$$\overset{\circ}{t}_{AB} := t_{AB} - \frac{1}{2}\eta_{AB}\eta^{CD}t_{CD}. \quad (3.30)$$

In particular, we denote the traceless part of  $h_{AB}$  by

$$\chi_{AB} \equiv \overset{\circ}{h}_{AB} := h_{AB} - \frac{1}{2}\eta_{AB}H, \quad (3.31)$$

where  $H := \eta^{AB}h_{AB}$ .

From (3.29) we get the gauge transformation for  $\chi_{AB}$ :

$$\chi_{AB} \rightarrow \chi_{AB} + \xi_{A||B} + \xi_{B||A} - \eta_{AB}\xi^C{}_{||C}.$$

Let us notice that gauge of  $\chi_{AB}$  depends only on  $\xi_A$ , hence it is not dependent on the part which is tangent to  $M_2$ .

The ten components of the tensor  $h_{\mu\nu}$  split naturally with respect to the  $2+2$ -splitting of the spacetime  $M = S^2 \times M_2$  into:

- components  $h_{ab}$  in  $M_2$ ,
- components  $h_{AB}$  on  $S^2$ ,
- mixed components  $h_{aA}$ .

However, for the description of the two degrees of freedom of the gravitational field one can divide ten components of the tensor  $h_{\mu\nu}$  differently, into axial and polar part. They split as follows:

- 7 polar components:  $h_{ab}$ ,  $h_a{}^A{}_{||A}$ ,  $\chi^A{}_{B||AC}\eta^{BC}$ ,  $H$ ;
- 3 axial components:  $h_{aA||B}\varepsilon^{AB}$ ,  $\chi^A{}_{B||AC}\varepsilon^{BC}$ .

In this section we shall consider only axial part of the gravitational field. In particular, the gauge transformation for the axial components of the tensor  $h_{\mu\nu}$  reduces to:

$$h_{aB||C}\varepsilon^{BC} \rightarrow h_{aB||C}\varepsilon^{BC} + (\xi_{B||C}\varepsilon^{BC})_{\ddagger a}, \quad (3.32)$$

$$\chi_A{}^B{}_{||BC}\varepsilon^{AC} \rightarrow \chi_A{}^B{}_{||BC}\varepsilon^{AC} + \frac{(\overset{\circ}{\Delta}+2)}{r^2}(\xi_{B||C}\varepsilon^{BC}) \quad (3.33)$$

(see also Appendix B). Let us notice that the gauge of all axial components depends only on  $\xi_A$  so they are not dependent on the infinitesimal change of coordinates on  $M_2$ . Moreover, we shall see in the sequel that using appropriate operators on  $S^2$  one can produce from  $h_{aB||C}\varepsilon^{BC}$  and  $\chi_A{}^B{}_{||BC}\varepsilon^{AC}$  a gauge invariant quantity.

### 3.3 Gauge invariants

The following object

$$\mathbf{y}_a := (\overset{\circ}{\Delta} + 2)(h_{aA||C}\varepsilon^{AC}) - (r^2\chi_A{}^B{}_{||BC}\varepsilon^{AC})_{\dagger a}. \quad (3.34)$$

is invariant with respect to the gauge transformation described in the previous subsection. It is useful to introduce another object (cf. [14]):

$$\mathbf{y} := r^2(h_{bA||C}\varepsilon^{AC})_{\dagger a}\varepsilon^{ab}. \quad (3.35)$$

which is also gauge independent. Let us notice that definitions (3.34) and (3.35) are the same as formulae (2.3) and (2.4) respectively. This means that for axial and invariants there are no “background mass” corrections. This phenomena is no longer valid for polar invariant  $\mathbf{x}$  (see [14]). The explicit proof of gauge invariance property for  $\mathbf{y}_a$  and  $\mathbf{y}$  is given in appendix B. The invariants  $\mathbf{y}_a$  and  $\mathbf{y}$  (introduced also in [14]) are not independent but they fulfill the following identity

$$r^2\mathbf{y}_{a\dagger b}\varepsilon^{ba} = (\overset{\circ}{\Delta} + 2)\mathbf{y} \quad (3.36)$$

which is a straightforward consequence of the above definitions of the objects. The dipole part of  $\mathbf{y}$  corresponds to the stationary solution of the field equations<sup>6</sup> (see [14]). We shall verify this property in the sequel analyzing asymptotics of the solutions at future null infinity  $\mathcal{I}^+$ .

### 3.4 Field equations

Let us consider linearized vacuum Einstein equations

$$r_{\nu\rho} := \frac{1}{2}(h^\sigma{}_{\rho;\nu\sigma} + h^\sigma{}_{\nu;\rho\sigma} - h_{\nu\rho}{}^{;\sigma}{}_\sigma - h^\sigma{}_{\sigma;\nu\rho} + h^\sigma{}_\alpha C^\alpha{}_{\nu\sigma\rho}) = 0. \quad (3.37)$$

The axial part of linearized Einstein equations on a Schwarzschild background can be easily described in terms of invariants [10] (we present the details in appendix D)

$$\varepsilon_d{}^a(r^2\mathbf{y})_{\dagger a} + r^2\mathbf{y}_d = -2r^4r_{Bd||D}\varepsilon^{BD} = 0, \quad (3.38)$$

$$(r^2\mathbf{y}^a)_{\dagger a} = 2r^4r_A{}^B{}_{||BC}\varepsilon^{AC} = 0. \quad (3.39)$$

The above equations take the same form as in the case of the flat background<sup>7</sup>. Moreover, they imply a second order hyperbolic equation for  $\mathbf{y}$ :

$$(r^{-2}(r^2\mathbf{y})_{\dagger a})_{\dagger a} + r^{-2}(\overset{\circ}{\Delta} + 2)\mathbf{y} = 0.$$

However, this equation is no longer a wave operator but Regge-Wheeler equation

$$(\square + \frac{8m}{r^3})\mathbf{y} = 0. \quad (3.40)$$

## 4 Gauge transformation of linearized Riemann tensor and axial invariants

In [10] one can find details of calculations which we would like to present in this section. Those calculations are related to the gauge transformations of  $r^\mu{}_{\nu\rho\sigma} := \delta R^\mu{}_{\nu\rho\sigma}$ ,  $\delta R_{\mu\nu\rho\sigma}$  and  $r_{\mu\nu} := \delta R_{\mu\nu}$  (see appendix A). The linearized Ricci  $r_{\mu\nu}$  is obviously gauge invariant but  $r_{\mu\nu\rho\sigma}$  and  $\delta R_{\mu\nu\rho\sigma}$  are in general gauge dependent.

<sup>6</sup>We shall introduce them in the next subsection.

<sup>7</sup>This phenomenon is valid *only* for axial part.

Using the formula (3.26) one can show that the gauge transformation of the linearized Riemann tensor (A.7) discussed in appendix A takes the following form:

$$\begin{aligned} \delta R_{\mu\nu\rho\sigma} \rightarrow & \delta R_{\mu\nu\rho\sigma} + \xi_\alpha (R^\alpha_{\mu\sigma\rho;\nu} + R^\alpha_{\sigma\mu\nu;\rho} + R^\alpha_{\nu\rho\sigma;\mu} + R^\alpha_{\rho\nu\mu;\sigma}) + \\ & 2\xi_{\alpha;\mu} R^\alpha_{\nu\rho\sigma} + 2\xi_{\alpha;\rho} R^\alpha_{\sigma\mu\nu} + 2\xi_{\alpha;\sigma} R^\alpha_{\rho\nu\mu} + 2\xi_{\alpha;\nu} R^\alpha_{\mu\sigma\rho}. \end{aligned} \quad (4.1)$$

Let us denote the gauge term as

$$\delta R_{\mu\nu\rho\sigma} \rightarrow \delta R_{\mu\nu\rho\sigma} + \text{gage}(\delta R_{\mu\nu\rho\sigma})$$

hence the corresponding components take the form:

$$\text{gage}(\delta R_{abED}) = 0, \quad (4.2)$$

$$\text{gage}(\delta R_{abeD}) = \frac{2m}{r^3} (\xi_b \eta_{ae} - \xi_a \eta_{be})_{||D} + \frac{m}{r} \left( \eta_{be} \left( \frac{\xi_D}{r^2} \right)_{,a} - \eta_{ae} \left( \frac{\xi_D}{r^2} \right)_{,b} \right), \quad (4.3)$$

$$\begin{aligned} \text{gage}(\delta R_{ABDd}) &= \frac{m}{r^3} (\xi_d ||_A \eta_{BD} - \xi_d ||_B \eta_{AD}) + \frac{2m}{r^3} (\xi_B \eta_{DA} - \xi_A \eta_{DB})_{\dagger d} \\ &+ \frac{4m\sqrt{k}}{r^4} \varepsilon_d (\xi_B \eta_{DA} - \xi_A \eta_{DB}), \end{aligned} \quad (4.4)$$

$$\text{gage}(\delta R_{AbCd}) = -\frac{m}{r^3} (\eta_{bd} \xi_{A||C} + \eta_{db} \xi_{C||A}) + \eta_{AC} F(\xi)_{db}, \quad (4.5)$$

where

$$F(\xi)_{db} := \frac{m}{r^2} \left[ \left( \frac{\xi_d}{r} \right)_{\dagger b} + \left( \frac{\xi_b}{r} \right)_{\dagger d} + \frac{\sqrt{k}}{2r^2} (\xi_d \varepsilon_b + \xi_b \varepsilon_d) - 2\eta_{bd} \frac{\sqrt{k}}{r} \varepsilon^a \xi_a \right].$$

To construct axial part of the Riemann tensor  $\delta R_{\mu\nu\rho\sigma}$  we use some “spherical operators” and obtain the following gauge dependence:

$$r^2 \delta R_{abeD||A} \varepsilon^{DA} \rightarrow r^2 \delta R_{abeD||A} \varepsilon^{DA} + \frac{m}{r^3} (\eta_{be} (\xi_{D||A} \varepsilon^{DA})_{,a} - \eta_{ae} (\xi_{D||A} \varepsilon^{DA})_{,b}), \quad (4.6)$$

$$\delta R_{AB}{}^C{}_{d||C} \varepsilon^{AB} \rightarrow \delta R_{AB}{}^C{}_{d||C} \varepsilon^{AB} - \frac{4m}{r^3} (\xi_{A||B} \varepsilon^{AB})_{\dagger d}, \quad (4.7)$$

$$\overset{\circ}{\delta} R_{AbBd} \rightarrow \overset{\circ}{\delta} R_{AbBd} - \frac{m}{r^3} (\xi_{A||B} + \xi_{B||A} - \eta_{AB} \xi^D{}_{||D}) \eta_{bd}, \quad (4.8)$$

$$\delta R_{abED} \rightarrow \delta R_{abED}. \quad (4.9)$$

Let us notice that the components of the linearized Riemann tensor (4.6–4.8) may be “corrected” in such a way that we obtain gauge-independent objects:

$$\begin{aligned} r^2 (\overset{\circ}{\Delta} + 2) \delta R_{abcD||A} \varepsilon^{DA} \varepsilon^{ab} + \frac{4m}{r} \varepsilon^b{}_c (r^2 \chi_A{}^B{}_{||BC} \varepsilon^{AC})_{\dagger b} = \\ r^2 (\overset{\circ}{\Delta} + 2) r_{abcD||A} \varepsilon^{DA} \varepsilon^{ab} + \frac{4m}{r} \varepsilon^b{}_c (r^2 \chi_A{}^B{}_{||BC} \varepsilon^{AC})_{\dagger b} + \frac{m}{r^3} h_{Da||A} \eta_{bc} \varepsilon^{DA} \\ \delta R_{AB}{}^C{}_{d||C} \varepsilon^{AB} + \frac{4m}{r^3} h_{dB||C} \varepsilon^{BC} = r_{AB}{}^C{}_{d||C} \varepsilon^{AB} + \frac{5m}{r^3} h_{dB||C} \varepsilon^{BC} \end{aligned}$$

$$\overset{\circ}{\delta}R_{AbBd} + \frac{m}{r^3}\eta_{bd}\chi_{AB} = \overset{\circ}{r}_{AbBd} + \frac{2m}{r^3}\eta_{bd}\chi_{AB}$$

$$\delta R_{abED} = r_{abED}$$

For completeness, we give here also the above above formulae in terms of the “true” linearized Riemann tensor  $r_{\mu\nu\rho\sigma}$  which is related to  $\delta R_{\mu\nu\rho\sigma}$  by formula (A.4). More precisely, the relation (A.4) for various components of Riemann tensor gives the following:

$$\begin{aligned} r_{abED} &= \delta R_{abED}, \quad r_{AbED} = \delta R_{AbED}, \quad r_{ABEd} = \delta R_{ABEd} - \frac{m}{r^3}h_{Ad}\eta_{BE} \\ r_{abeD} &= \delta R_{abeD} - \frac{m}{r^3}h_{Da}\eta_{be}, \quad r_{AbBd} = \delta R_{AbBd} - \frac{m}{r^3}h_{AB}\eta_{bd}. \end{aligned}$$

One can easily verify that the attempt to exchange  $\delta R_{\mu\nu\lambda\rho}$  with  $r_{\mu\nu\lambda\rho}$  does not improve the invariance of (4.6 – 4.8). In particular, the “metric corrections” although in a different form will still occur. We prefer to use the tensor  $\delta R_{\mu\nu\lambda\rho}$  because it possesses all the symmetries of the usual full curvature tensor.

The explicit formulae for the axial part of  $\delta R_{\mu\nu\rho\sigma}$  in terms of the metric  $h_{\mu\nu}$

$$\begin{aligned} 2\delta R_{abED} &= h_{aD\ddagger b||E} + h_{bE\ddagger a||D} - h_{aE\ddagger b||D} - h_{bD\ddagger a||E} + \\ &+ \frac{\sqrt{k}}{r}\varepsilon_b(2h_{aD||E} - 2h_{aE||D}) + \frac{\sqrt{k}}{r}\varepsilon_a(2h_{bE||D} - 2h_{bD||E}) \end{aligned} \quad (4.10)$$

$$\begin{aligned} 2\delta R_{ABEd} &= h_{Ad||BE} - h_{Bd||AE} + h_{BE||A\ddagger d} - h_{AE||B\ddagger d} + \\ &+ \frac{\sqrt{k}}{r}(2h_{BE||A}\varepsilon_d - 2h_{AE||B}\varepsilon_d + h_{ad||A}\varepsilon^a\eta_{BE} - h_{ad||B}\varepsilon^a\eta_{AE}) + \\ &+ \frac{\sqrt{k}}{r}(h_{Bd\ddagger a}\varepsilon^a\eta_{AE} - h_{Ad\ddagger a}\varepsilon^a\eta_{BE} + h_{Aa\ddagger d}\eta_{EB}\varepsilon^a - h_{Ba\ddagger d}\eta_{EA}\varepsilon^a) + \\ &+ 2\frac{k}{r^2}\varepsilon^a\varepsilon_d(h_{Aa}\eta_{EB} - h_{Ba}\eta_{EA}) \end{aligned} \quad (4.11)$$

$$\begin{aligned} 2\delta R_{abeD} &= h_{aD\ddagger be} - h_{ae\ddagger b||D} - h_{bD\ddagger ae} + h_{be\ddagger a||D} + \\ &+ \frac{\sqrt{k}}{r}(h_{aD\ddagger e}\varepsilon_b - h_{bD\ddagger e}\varepsilon_a + h_{De\ddagger a}\varepsilon_b - h_{De\ddagger b}\varepsilon_a + h_{be||D}\varepsilon_a - h_{ae||D}\varepsilon_b) \end{aligned} \quad (4.12)$$

$$\begin{aligned} 2\delta R_{AbBd} &= h_{Ad\ddagger b||B} - h_{bd||AB} - h_{AB\ddagger bd} + h_{bB||A\ddagger d} + \\ &+ \frac{\sqrt{k}}{r}(h_{Ad||B}\varepsilon_b + h_{bB||A}\varepsilon_d - h_{bA||B}\varepsilon_d - h_{Bd||A}\varepsilon_b) + \\ &- \frac{\sqrt{k}}{r}(h_{AB\ddagger b}\varepsilon_d + h_{AB\ddagger d}\varepsilon_b + h_{fd\ddagger b}\varepsilon^f\eta_{AB} + h_{bf\ddagger d}\varepsilon^f\eta_{AB} + \\ &+ h_{bd\ddagger f}\varepsilon^f\eta_{AB}) + \frac{m}{r^3}(h_{AB}\varepsilon_b\varepsilon_d - h_{AB}\eta_{db} - 2h_{db}\eta_{AB}) \end{aligned} \quad (4.13)$$

enables one to express them in terms of the invariants  $\mathbf{y}_a$  and  $\mathbf{y}$ :

$$\frac{1}{2}r^2\varepsilon^{ab}\varepsilon^{ED}\delta R_{abED} = \mathbf{y} \quad (4.14)$$

$$\begin{aligned} r^2(\overset{\circ}{\Delta} + 2)\delta R_{AB}{}^C{}_{d||C}\varepsilon^{AB} + \frac{4m}{r^3}(r^2\chi_A{}^B{}_{||BC}\varepsilon^{AC})_{\ddagger d} &= \frac{1}{r}\varepsilon_d{}^a(\overset{\circ}{\Delta} + 2)(r\mathbf{y})_{\ddagger a} - \frac{4m}{r^5}\varepsilon_d{}^a(r^2\mathbf{y})_{\ddagger a} \\ &= (\overset{\circ}{\Delta} + 2)\mathbf{y}_d + r\sqrt{k}\varepsilon^a(\mathbf{y}_{a\ddagger d} - \mathbf{y}_{d\ddagger a}) - \frac{4m}{r^3}\mathbf{y}_d \end{aligned} \quad (4.15)$$

$$4r^4 \left[ \delta R^A{}_{bBd||AD} \varepsilon^{BD} + \frac{m}{r^3} \eta_{bd} \chi_A{}^B{}_{||BC} \varepsilon^{AC} \right] = (r^2 \mathbf{y}_d)_{\dagger b} + (r^2 \mathbf{y}_b)_{\dagger d} \quad (4.16)$$

$$\begin{aligned} r^2(\overset{\circ}{\Delta} + 2) \delta R_{abeD||A} \varepsilon^{DA} \varepsilon^{ab} + \frac{4m}{r} \varepsilon_e{}^b (r^2 \chi_A{}^B{}_{||BC} \varepsilon^{AC})_{\dagger b} = \\ = \frac{1}{r} [r(\overset{\circ}{\Delta} + 2) \mathbf{y}]_{\dagger e} + \frac{2m}{r^3} (r^2 \mathbf{y})_{\dagger e} . \end{aligned} \quad (4.17)$$

The above formulae describe the relation between axial part of the linearized Riemann tensor (corrected to gauge-independent form) and our standard gauge-independent quantities  $\mathbf{y}$ ,  $\mathbf{y}_a$ . The equations (4.14–4.17) are generalizations of the formulae (2.11).

## 5 Asymptotics for solutions of Regge-Wheeler equation at null infinity

Let us consider Regge-Wheeler equation (3.40) in null coordinates  $(u, v)$

$$-\frac{2}{k} [\partial_u (r^2 \partial_v \mathbf{y}) + \partial_v (r^2 \partial_u \mathbf{y})] + \overset{\circ}{\Delta} \mathbf{y} + \frac{8m}{r^3} \mathbf{y} = 0 \quad (5.1)$$

and let us assume (cf. [8]) that the solution of (5.1) is in the following asymptotic form (see appendix C):

$$\mathbf{y} = \sum_{n=1}^5 \frac{a_n(u, \theta, \phi)}{r^n} + O\left(\frac{1}{r^6}\right) \quad (5.2)$$

where  $a_n$  are functions of  $(u, \theta, \phi)$  hence they are well defined on  $\mathcal{I}^+$ . From the assumption that  $a_n$  do not depend on  $v$  we have for each  $n$ :

$$\partial_v a_n = 0.$$

Moreover, denote the  $u$ -derivative by dot e.g.

$$\dot{a}_n = \partial_u a_n.$$

Additionally, using (3.5) together with the definition of  $\varepsilon_a$  one can express the derivatives of  $k$  and  $r$

$$\partial_v k = \frac{m}{r^2} k, \quad \partial_u k = -\frac{m}{r^2} k, \quad \partial_v r = \frac{1}{2} k, \quad \partial_u r = -\frac{1}{2} k.$$

The series (5.2) inserted into equation (5.1) gives the following formula:<sup>8</sup>

$$\sum_{n=1}^5 \left[ n(n-1) \frac{a_n}{r^n} + \frac{2(4-n^2)a_n}{r^{n+1}} m + 2(n-1) \frac{\dot{a}_n}{r^{n-1}} + \frac{\overset{\circ}{\Delta} a_n}{r^n} \right] + O\left(\frac{1}{r^6}\right) = 0. \quad (5.3)$$

Comparing the coefficients at the same power of  $r$  we obtain recurrence relations for the coefficients  $a_n$ :

$$\begin{aligned} n=1 & \quad a_1 \text{— free data} \\ n=2 & \quad 2\dot{a}_2 + \overset{\circ}{\Delta} a_1 = 0 \\ n \geq 2 & \quad 2n\dot{a}_{n+1} + \left[ \overset{\circ}{\Delta} + n(n-1) \right] a_n - 2ma_{n-1}(n-3)(n+1) = 0 \end{aligned} \quad (5.4)$$

---

<sup>8</sup>We assume that the derivatives of the asymptotic terms  $O\left(\frac{1}{r^6}\right)$  are at least of the same asymptotic order.



Let us rewrite equation (5.4) using new integer parameter  $l := (n - 1)$ :

$$2(l+1)\dot{a}_{l+2} = - \left[ \overset{\circ}{\Delta} + l(l+1) \right] a_{l+1} + 2ma_l(l-2)(l+2) \quad (5.5)$$

**Remark.** The Remark from Subsection 2.7 about NP constants remains valid for the Schwarzschild background. More precisely, the right-hand side of (5.5) vanishes on the spherical harmonics subspace corresponding to  $l = 2$  and quadrupole part of  $a_4$  does not depend on  $u$ .

### 5.1 “Peeling” for the axial part of Weyl tensor

We continue our asymptotic considerations based on the assumption (5.2), in particular the first term of the asymptotics gives

$$\mathbf{y} = \mathcal{O}\left(\frac{1}{r}\right).$$

The equation (3.38) written in an equivalent form:

$$\mathbf{y}_d = -\frac{1}{r^2}\varepsilon_d{}^a(r^2\mathbf{y})_{\dagger a} \quad (5.6)$$

may be used to obtain the asymptotic behaviour of  $\mathbf{y}_a$ :

$$\begin{aligned} (r^2\mathbf{y})_{\dagger u} &= r\dot{a}_1 + (\dot{a}_2 - \frac{1}{2}a_1) + \frac{1}{r}(\dot{a}_3 + ma_1) + \frac{1}{r^2}(\dot{a}_4 + \frac{1}{2}a_3) + \\ &\quad \frac{1}{r^3}(\dot{a}_5 - ma_3 + a_4) + \mathcal{O}\left(\frac{1}{r^4}\right) \end{aligned} \quad (5.7)$$

$$(r^2\mathbf{y})_{\dagger v} = -\frac{1}{2}a_1 + \frac{1}{r}ma_1 + \frac{1}{2r^2}a_3 + \frac{1}{r^3}(-ma_3 + a_4) + \mathcal{O}\left(\frac{1}{r^4}\right) \quad (5.8)$$

or more explicitly, using formula (5.6) we obtain asymptotics of both components of  $\mathbf{y}_a$ :

$$\mathbf{y}_u = -\frac{1}{r^2}\varepsilon_u{}^u(r^2\mathbf{y})_{\dagger u} \sim -\frac{\dot{a}_1}{r} \quad (5.9)$$

$$\mathbf{y}_v = -\frac{1}{r^2}\varepsilon_v{}^v(r^2\mathbf{y})_{\dagger v} \sim -\frac{a_1}{2r^2}. \quad (5.10)$$

From (4.17) one can show

$$r(\overset{\circ}{\Delta} + 2)\delta R_{abcd}{}_{||A}\varepsilon^{DA}\varepsilon^{ab} + \frac{4m}{r^2}\varepsilon_c{}^b(r^2\chi_A{}^B{}_{||BC}\varepsilon^{AC})_{,b} = \frac{1}{r^2}(\overset{\circ}{\Delta} + 2)(r\mathbf{y})_{,c} + \frac{2m}{r^4}(r^2\mathbf{y})_{,c}. \quad (5.11)$$

From asymptotics (5.2) for the solutions of the equation (3.40) we conclude that the  $v$ -component of equation (5.11) has the following asymptotic behaviour

$$\frac{1}{r^2}(\overset{\circ}{\Delta} + 2)(r\mathbf{y})_{,v} + \frac{2m}{r^4}(r^2\mathbf{y})_{,v} = \mathcal{O}\left(\frac{1}{r^4}\right),$$

because  $(r\mathbf{y})_{,v} \sim \left(\frac{a_2}{r}\right) = \mathcal{O}\left(\frac{1}{r^2}\right)$  and by the use of (5.8). Moreover, the asymptotic condition  $\partial_v\chi^A{}_B = \mathcal{O}\left(\frac{1}{r^2}\right)$ , which is usually fulfilled for the asymptotically flat metric, implies

$$\frac{4m}{r^2}\varepsilon_v{}^b(r^2\chi_A{}^B{}_{||BC}\varepsilon^{AC})_{,b} = \mathcal{O}\left(\frac{1}{r^4}\right).$$

Hence

$$\delta\Psi_1^{\text{NP}} \rightarrow r\delta R_{abvD}||_A \varepsilon^{DA} \varepsilon^{ab} = \mathcal{O}\left(\frac{1}{r^4}\right) \quad (5.12)$$

gives a usual “peeling” because the gauge-dependent term

$$\frac{4m}{r^2} \varepsilon_v{}^v (r^2 \chi_A{}^B ||_{BC} \varepsilon^{AC})_{,v} = \mathcal{O}\left(\frac{1}{r^4}\right)$$

has the same asymptotics as the full invariant.

Similarly we may investigate the invariant

$$\left[ \overset{\circ}{\delta} R^A{}_{bBd} + \frac{m}{r^3} \eta_{bd} \chi^A{}_B \right] ||_{AC} \varepsilon^{BC} r^2 \quad (5.13)$$

which equals

$$\frac{1}{r^2} [(r^2 \mathbf{y}_d)_{\ddagger b} + (r^2 \mathbf{y}_b)_{\ddagger d}] \quad (5.14)$$

from (4.16). Using (5.2), (5.7–5.8) we obtain asymptotics  $\mathcal{O}(\frac{1}{r^5})$  for the term (5.14) if we assume that both indices  $b = d = v$ . Hence

$$\delta\Psi_0^{\text{NP}} \rightarrow r^2 \overset{\circ}{\delta} R_v{}^A{}_{vB} ||_{AC} \varepsilon^{BC} = \mathcal{O}\left(\frac{1}{r^5}\right) \quad (5.15)$$

Here the result is simpler because  $\overset{\circ}{\delta} R_{vAvB}$  is already gauge-independent ( $\eta_{vv} = 0$ ).

This way we have proved „peeling” for axial part of the Weyl tensor corresponding to Newman-Penrose scalars  $\Psi_0$  and  $\Psi_1$ . In a similar way one can check the asymptotics for the remaining three scalars  $\Psi_3$ ,  $\Psi_4$  and  $\Psi_2$ . This is a consequence of the same tensor equations (5.11, 5.13) but we need to use other components of them. The corresponding invariants have the following asymptotic behaviour:

$$\frac{1}{r^2} (\overset{\circ}{\Delta} + 2)(r\mathbf{y})_{,u} + \frac{2m}{r^4} (r^2 \mathbf{y})_{,u} = \mathcal{O}\left(\frac{1}{r^2}\right), \quad \frac{2}{r^2} (r^2 \mathbf{y}_u)_{\ddagger u} = \mathcal{O}\left(\frac{1}{r}\right).$$

It is relatively easy to verify that the gauge-dependent “correction” terms depending on  $\chi^A{}_B$  have the same asymptotic order and finally we obtain

$$\delta\Psi_3^{\text{NP}} \rightarrow r\delta R_{abuD}||_A \varepsilon^{DA} \varepsilon^{ab} = \mathcal{O}\left(\frac{1}{r^2}\right), \quad (5.16)$$

$$\delta\Psi_4^{\text{NP}} \rightarrow \overset{\circ}{\delta} R_u{}^A{}_{uB} ||_{AC} \varepsilon^{BC} r^2 = \mathcal{O}\left(\frac{1}{r}\right), \quad (5.17)$$

and for the last gauge-independent component

$$\delta\Psi_2^{\text{NP}} \rightarrow \varepsilon^{ab} \varepsilon^{ED} \delta R_{abED} = \mathcal{O}\left(\frac{1}{r^3}\right). \quad (5.18)$$

We would like to stress that asymptotic behaviour of  $\Psi_2, \Psi_3, \Psi_4$  given by (5.16–5.18) is not ambiguous<sup>9</sup>. However, in general,  $\Psi_0$  and  $\Psi_1$  may have weaker asymptotic behaviour [15]. It is clear from the above investigations that axial part obeys “strong peeling”, but this is completely not obvious that the same is true for the second degree of freedom described by the polar part.

<sup>9</sup>It is shown in [16], even for much weaker asymptotic assumptions — so called polyhomogeneous asymptotics (i.e. including terms  $W(\ln r)r^{-k}$  where  $W$ —polynomial), that the same asymptotics is valid for  $\Psi_2, \Psi_3, \Psi_4$  but for  $\Psi_0$  and  $\Psi_1$ , in general, we have terms of order  $r^{-4} \ln r$ .

The results of this section we summarize in the following table:

Weyl	C – K – N	axial invariant	N – P	asymptotics
$\overset{\circ}{W}_v{}^A{}_{vB}$	$r^2\alpha_{\text{CKN}}$	$r^{-2}(r^2\mathbf{y}_v)_{\ddagger v}$	$\Psi_0$	$r^{-5} \ (r^{-3.5})$
$\overset{\circ}{W}_u{}^A{}_{uB}$	$r^2\underline{\alpha}_{\text{CKN}}$	$r^{-2}(r^2\mathbf{y}_u)_{\ddagger u}$	$\Psi_4$	$r^{-1}$
$r\varepsilon^{ab}W_{abv}{}^A$	$r\beta_{\text{CKN}}$	$r^{-2}(\overset{\circ}{\Delta} + 2)(r\mathbf{y})_{,v} + \frac{2m}{r^4}(r^2\mathbf{y})_{,v}$	$\Psi_1$	$r^{-4} \ (r^{-3.5})$
$r\varepsilon^{ab}W_{abu}{}^A$	$r\underline{\beta}_{\text{CKN}}$	$r^{-2}(\overset{\circ}{\Delta} + 2)(r\mathbf{y})_{,u} + \frac{2m}{r^4}(r^2\mathbf{y})_{,u}$	$\Psi_3$	$r^{-2}$
$\varepsilon^{ab}\varepsilon^{ED}W_{abED}$	$\sigma_{\text{CKN}}$	$r^{-2}\mathbf{y}$	$\Psi_2$	$r^{-3}$

Here  $W$  corresponds to  $\delta R$ , and in brackets we give the asymptotic results of Christodolou-Klainerman-Nicoló (cf. [15]).

## 6 Conclusions

In [20] one can find the following statement: “For even waves, it has not yet been possible to derive an equation like Regge-Wheeler from the perturbed NP equations”. This question has been resolved in [7] but in our opinion not in a satisfactory way (see the discussion at the end of this Section). We shall explain in a separate paper why in [20] one can easily formulate Regge-Wheeler equation for odd degree of freedom in terms of Newman-Penrose scalars but for polar (even) degree of freedom it was difficult to formulate Zerilli equation in terms of Newman-Penrose quantities. Although decoupled equations do exist for  $\Psi_2$  and  $\Psi_{-2}$  [22, 1], even on a Kerr background, they are not deformations<sup>10</sup> of the usual wave equations in the asymptotic region. Moreover, the NP special null tetrad (chosen in [20] and [1]) is not symmetric with respect to the interchange of null coordinates  $u$  and  $v$ . In other words it has to be chosen in a different way close to future ( $\mathcal{I}^+$ ) and past ( $\mathcal{I}^-$ ) null infinity. We would like to convince the reader that the Teukolsky equation for  $\Psi_{-2}^{\text{Price}} = \Psi_4^{\text{NP}}$  is not a primary equation describing gravitational waves in asymptotic region (see also [17] for the review of the Teukolsky formalism). The reasons are the following:

- The Teukolsky equation is not a deformation of a d’Alembert equation in contrast to the Regge-Wheeler and Zerilli.
- The initial data on a slice  $t = \text{const.}$ , instead of position and momenta  $(\mathbf{x}, \dot{\mathbf{x}})$ , corresponds rather to second and third time derivatives of the position  $(\frac{\partial^2}{\partial t^2}\mathbf{x}, \frac{\partial^3}{\partial t^3}\mathbf{x})$ .

We may think about this equation as an evolution equation for the acceleration  $\ddot{\mathbf{x}}$ . Usually, when we use Fourier transform technique, it is not so important for plane waves which variable  $(\mathbf{x}$  or  $\ddot{\mathbf{x}})$  we are using as a canonical position. However, choosing  $\ddot{\mathbf{x}}$  we exclude from the beginning all stationary solutions which are also physically important for the wave operator.

In our gauge invariant quasilocal formalism we checked the “peeling property” for the various components of the linearized Weyl tensor. In this paper we calculated this property only for the axial degree of freedom (governed by the Regge-Wheeler equation) and we showed that it is valid in its original strong form similar to the case of a flat background (see the table in subsection 2.6). However, for polar degree of freedom (governed by Zerilli equation [23], [14]) we may have some obstructions. More precisely, because NP scalars are not gauge-invariant one could possibly “damage” their asymptotics via gauge transformations. This is related to a more complicated behaviour of the asymptotic solutions of the Zerilli equation and not obvious asymptotics of the gauge transformations. Although we showed that for the axial part we could not “damage” asymptotics

<sup>10</sup>By deformation of the wave operator we mean a hyperbolic equation such that when parameter  $m$  vanishes it becomes the usual wave equation.

via gauge transformation, it is not evident that the same is true for the polar degrees of freedom. We shall elaborate upon this issue in a separate paper.

We believe that peeling phenomena for linearized gravitational field is a simpler property for the gauge-invariants substituting NP scalars than for NP scalars themselves.

We also hope that our results can be applied for improving Christodoulou-Klainerman-Nicolò [15] asymptotics on  $\mathcal{I}^+$  for nonlinear Cauchy data “sufficiently close” to Schwarzschild.

It is not easy to relate the results of [7] with our approach. The authors do not give explicit formulae for their  $\hat{\Psi}_k$ . However, the deformation of  $\Psi_2^{\text{NP}}$  proposed by them differs from ours. In our case real and imaginary part of “deformed”  $\Psi_2^{\text{NP}}$  do not fulfill the same equation like in [7]. Moreover, the equations (3.77-3.81) on p. 847 suggest that their invariants  $\hat{\Psi}_2, \hat{\Psi}'_2$  are quasilocally related to  $\Psi_4^{\text{NP}}$  and  $\Psi_0^{\text{NP}}$ . We would like to stress that our invariants appear as the “natural” positions in the symplectic analysis. The hamiltonian approach enables one to derive the Regge-Wheeler equation, resp. the Zerilli equation, as an Euler-Lagrange second order equation for the axial, respectively polar, part of the field (see [14]). This suggests that a deformed  $\Re\Psi_2^{\text{NP}}$  should fulfill the Zerilli equation but  $\Im\Psi_2^{\text{NP}}$  fulfills Regge-Wheeler equation.

## A Linearization of Riemann tensor on vacuum background

To fix the notation and for completeness we present in this appendix the standard linearization formulae. Let  $M$  be a spacetime with pseudoriemannian metric  $g_{\mu\nu}$ . We define a linear perturbation of the metric as

$$h_{\mu\nu} \equiv \delta g_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu} . \quad (\text{A.1})$$

where by  $\eta_{\mu\nu}$  we denote the background metric. The tensor  $h_{\mu\nu}$  is often called perturbation of the metric  $\eta_{\mu\nu}$ . For the inverse metric the perturbation has opposite sign:

$$\delta g^{\mu\nu} = -h^{\mu\nu} := -\eta^{\mu\lambda}\eta^{\nu\kappa}h_{\lambda\kappa} .$$

Similarly, we have linearized Christoffel symbols (which are tensors):

$$\delta\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}\eta^{\mu\kappa}(h_{\kappa\nu;\lambda} + h_{\kappa\lambda;\nu} - h_{\lambda\nu;\kappa}) , \quad (\text{A.2})$$

where as usual all manipulations are with respect to the background metric, background connection etc. We have also linearized Riemann tensor:

$$\delta R^\alpha{}_{\beta\mu\nu} = \delta\Gamma^\alpha{}_{\beta\nu;\mu} - \delta\Gamma^\alpha{}_{\beta\mu;\nu} . \quad (\text{A.3})$$

We would like to stress that in general for curved background the linearization of the Riemann tensor depends on the position of indices. Let us denote by  $r^\alpha{}_{\beta\mu\nu} := \delta R^\alpha{}_{\beta\mu\nu}$  the linearization of the Riemann tensor with one upper index, and

$$r_{\kappa\beta\mu\nu} := r^\alpha{}_{\beta\mu\nu}\eta_{\alpha\kappa} .$$

Let us compare  $r_{\kappa\beta\mu\nu}$  with the corresponding linearization of the curvature tensor with all indices lowered (i.e.  $\delta R_{\alpha\nu\rho\mu} = \delta(g_{\alpha\kappa}R^\kappa{}_{\nu\rho\mu})$ ). The relation between  $\delta R_{\alpha\nu\rho\mu}$  and  $r_{\alpha\nu\rho\mu}$  takes the form

$$\delta R_{\alpha\nu\rho\mu} := R_{\alpha\nu\rho\mu}(g) - \mathcal{R}_{\alpha\nu\rho\mu}(\eta) = r_{\alpha\nu\rho\mu} + h_{\alpha\sigma}\mathcal{R}^\sigma{}_{\nu\rho\mu} , \quad (\text{A.4})$$

where by  $\mathcal{R}^\sigma{}_{\nu\rho\mu}$  we have denoted curvature tensor of the background metric.

We would like to stress that tensor  $r_{\alpha\nu\rho\mu}$  has not all symmetries of the usual curvature tensor. This unpleasant property may be verified in the formulae below.

From definition of  $r_{\mu\nu\sigma\rho}$  and equation (A.2) we can express (A.3) in terms of  $h_{\mu\nu}$  as follows

$$r_{\mu\nu\sigma\rho} = \frac{1}{2}(h_{\mu\nu;\sigma\rho} - h_{\mu\nu;\sigma\rho} + h_{\mu\rho;\nu\sigma} - h_{\nu\rho;\mu\sigma} - h_{\mu\sigma;\nu\rho} + h_{\nu\sigma;\mu\rho}) . \quad (\text{A.5})$$

The basic property of the curvature

$$h_{\nu\sigma;\mu\rho} - h_{\nu\sigma;\rho\mu} = h_{\alpha\sigma} R^\alpha{}_{\nu\mu\rho} + h_{\nu\alpha} R^\alpha{}_{\sigma\mu\rho} \quad (\text{A.6})$$

implies

$$2r_{\mu\nu\sigma\rho} = h_{\alpha\sigma} \mathcal{R}^\alpha{}_{\nu\mu\rho} + h_{\alpha\nu} \mathcal{R}^\alpha{}_{\sigma\mu\rho} + h_{\mu\rho;\nu\sigma} - h_{\nu\rho;\mu\sigma} - h_{\mu\sigma;\nu\rho} + h_{\nu\sigma;\mu\rho}. \quad (\text{A.7})$$

Let us denote by  $r_{\nu\rho}$  a linearization of the Ricci tensor defined as follows

$$r_{\nu\rho} := r^\sigma{}_{\nu\sigma\rho} = \eta^{\sigma\mu} r_{\mu\nu\sigma\rho}. \quad (\text{A.8})$$

If we assume that the background metric is Ricci flat (i.e. is a solution of vacuum Einstein equations)

$$\mathcal{R}_{\mu\nu} = 0,$$

then we have the following form for the linearized Ricci tensor:

$$r_{\nu\rho} = \frac{1}{2} (h_{\alpha\sigma} \mathcal{R}^\alpha{}_{\nu}{}^\sigma{}_{\rho} + h^\sigma{}_{\rho;\nu\sigma} + h^\sigma{}_{\nu;\rho\sigma} - h_{\nu\rho}{}^{;\sigma}{}_{\sigma} - h^\sigma{}_{\sigma;\nu\rho}). \quad (\text{A.9})$$

## B Gauge invariance of $\mathbf{y}$ , $\mathbf{y}_a$

Let us begin with the gauge transformation (3.28) for  $h_{aB}$ :

$$h_{aB} \rightarrow h_{aB} + \xi_{a,B} + \xi_{B,a} + 2\frac{\sqrt{k}}{r}\varepsilon_a\xi_B.$$

and let us denote by  $z_a := h_{aB||C}\varepsilon^{BC}$  a gauge-dependent axial part of  $h_{aB}$ . It is easy to check that  $z_a$  transforms as follows:

$$z_a \rightarrow z_a + (\xi_{B||C}\varepsilon^{BC})_{,a}. \quad (\text{B.1})$$

To compensate the gauge term  $(\xi_{B||C}\varepsilon^{BC})_{,a}$  let us analyze the gauge transformation for the axial part of  $\chi_{AB} = h_{AB} - \frac{1}{2}\eta_{AB}h_{CD}\eta^{CD}$ :

$$\chi_A{}^B{}_{||BC}\varepsilon^{AC} \rightarrow \chi_A{}^B{}_{||BC}\varepsilon^{AC} + \eta^{EB}(\xi_{A||EBC} + \xi_{E||ABC})\varepsilon^{AC}. \quad (\text{B.2})$$

Using the commutation relation for the second covariant derivatives

$$\xi_{A||EBC} = \xi_{A||ECB} + \xi_{F||E}\overset{2}{R}{}^F{}_{ABC} + \xi_{A||F}\overset{2}{R}{}^F{}_{EBC}, \quad (\text{B.3})$$

where  $\overset{2}{R}{}^F{}_{ABC}$  is the curvature of  $S^2$  given by (3.18), from (B.2) and the identity  $\xi^E{}_{||EAC}\varepsilon^{AC} = 0$ , we obtain (3.33)

$$\chi_A{}^B{}_{||BC}\varepsilon^{AC} \rightarrow \chi_A{}^B{}_{||BC}\varepsilon^{AC} + \xi_{A||C}{}^E{}_{||E}\varepsilon^{AC} + \frac{2}{r^2}\xi_{A||C}\varepsilon^{AC},$$

or in equivalent form

$$r^2\chi_A{}^B{}_{||BC}\varepsilon^{AC} \rightarrow r^2\chi_A{}^B{}_{||BC}\varepsilon^{AC} + (\overset{\circ}{\Delta} + 2)(\xi_{A||C}\varepsilon^{AC}). \quad (\text{B.4})$$

Finally the gauge formulae (B.1) and (B.4) imply that  $\mathbf{y}_a = (\overset{\circ}{\Delta} + 2)z_a - (r^2\xi_A{}^B{}_{||BC}\varepsilon^{AC})_{,a}$  is gauge independent.

Similarly, we can show gauge invariance of  $\mathbf{y}$ . We shall derive the relation between  $\mathbf{y}$  and the component  $r_{abED}\varepsilon^{ED}\varepsilon^{ab}$  of the linearized Riemann tensor which is gauge independent as has been shown in Section 4. From (A.7) we obtain

$$\begin{aligned} 2r_{abED} &= h_{aD\ddagger b||E} + h_{bE\ddagger a||D} - h_{aE\ddagger b||D} - h_{bD\ddagger a||E} + \\ &\quad \frac{\sqrt{k}}{r}\varepsilon_b(2h_{aD||E} - 2h_{aE||D}) + \frac{\sqrt{k}}{r}\varepsilon_a(2h_{bE||D} - 2h_{bD||E}). \end{aligned} \quad (\text{B.5})$$

Moreover from  $(\varepsilon^{AB})_{\ddagger a} = \frac{2\sqrt{k}}{r}\varepsilon_a\varepsilon^{AB}$  (cf. (3.24)) we get

$$\begin{aligned} 2r_{abED}\varepsilon^{ED} &= (h_{aD||E}\varepsilon^{ED})_{\ddagger b} + (h_{bE||D}\varepsilon^{ED})_{\ddagger a} - (h_{aE||D}\varepsilon^{ED})_{\ddagger b} - (h_{bD||E}\varepsilon^{ED})_{\ddagger a} \\ &= 2(h_{bE||D}\varepsilon^{ED})_{\ddagger a} - 2(h_{aE||D}\varepsilon^{ED})_{\ddagger b}. \end{aligned} \quad (\text{B.6})$$

and finally

$$r_{abED}\varepsilon^{ED}\varepsilon^{ab} = 2(h_{bE||D}\varepsilon^{ED})_{\ddagger a}\varepsilon^{ab}$$

which implies the demanded relation for  $\mathbf{y}$

$$\mathbf{y} = \frac{1}{2}r^2r_{abED}\varepsilon^{ED}\varepsilon^{ab}.$$

This way we get gauge invariance of  $\mathbf{y}$  from gauge independence of  $r_{abED}$ . However, equation (3.36) also implies this result from the gauge invariance of  $\mathbf{y}_a$ .

## C Compactification of Regge–Wheeler equation near $\mathcal{I}^+$

The equation (5.1) in the coordinates  $(u, \rho, x^A)$  can be rewritten<sup>11</sup> in the following form:

$$2\partial_u\partial_\rho\Phi + \rho\partial_\rho[k\partial_\rho(\rho\Phi)] + \overset{\circ}{\Delta}\Phi + 8m\rho\Phi = 0 \quad (\text{C.1})$$

where  $\Phi := \rho^{-1}\mathbf{y}$  and  $k = 1 - 2m\rho$ . Standard arguments, using domain of dependence considerations together with conformal covariance of Equation (C.1), show that smooth initial data which are compactly supported on some Cauchy hypersurface for the Kruskal–Schwarzschild spacetime lead to the solutions of equation (C.1) such that the rescaled  $\Phi$  smoothly extends across  $\mathcal{I}^+$  (cf. [9]). This means that the assumption (5.2) is fulfilled for a large class of solutions for the Regge–Wheeler equation (5.1). On the other hand, no conditions on initial data which are not compactly supported are known, which would guarantee smoothness of solutions across  $\mathcal{I}^+$ .

## D Axial part of linearized Einstein equations

The equation (3.39) contains the component  $\overset{\circ}{r}{}^A{}_{B||AC}\varepsilon^{BC}$  where by  $\overset{\circ}{r}_{AB} \equiv TS(r_{AB})$  we denote traceless symmetric part of tensor  $r_{AB}$  (cf. (3.30))

$$\overset{\circ}{r}_{AB} := r_{AB} - \frac{1}{2}\eta_{AB}r_{CD}\eta^{CD} \equiv TS(r_{AB}), \quad TS(h_{AB}) \equiv \chi_{AB}.$$

The general formula (A.9) takes the following form for  $r_{AB}$  on Schwarzschild background:

$$\begin{aligned} 2r_{AB} &= h_{ab}C^a{}_A{}^b{}_B + h_{CD}C^C{}_A{}^D{}_B + h^a{}_{B;Aa} + h^F{}_{B;AF} + h^a{}_{A;Ba} + h^F{}_{A;BF} \\ &\quad + h_{AB;{}^a{}_a} - h_{AB;{}^F{}_F} - h_{;AB} \end{aligned} \quad (\text{D.1})$$

where  $h := \eta^{\mu\nu}h_{\mu\nu}$ .

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<sup>11</sup>We remind that  $u := t - r - 2m \ln(r - 2m)$ ,  $\rho := \frac{1}{r}$  and  $x^A$  are spherical angles.

Let us notice that the first two terms in (D.1) containing background Riemann  $C_{\mu\nu\lambda\kappa}$  are proportional to the metric  $\eta_{AB}$  so they disappear in  $\overset{\circ}{r}_{AB}$ . To give explicit expression for  $r_{AB}$  in terms of  $h_{\mu\nu}$  we need first to examine the terms  $h_{aB;Ab}$ ,  $h_{AB;CD}$  and  $h_{AB;ab}$ . Using extensively the formulae from Section 3 one can show that the following equalities hold:

$$\begin{aligned} h_{bB;Ad} &= h_{bB||A\ddagger d} + \frac{\sqrt{k}}{r}(h_{AB\ddagger d}\varepsilon_b + 2h_{bB||A}\varepsilon_d - \varepsilon^f h_{bf\ddagger d}\eta_{AB}) + \\ &\quad \frac{k}{r^2}(3h_{AB}\varepsilon_d\varepsilon_b + h_{bf}\varepsilon^f\varepsilon_d\eta_{AB}) + \frac{m}{r^3}(h_{bd}\eta_{AB} - h_{AB}\varepsilon_b\varepsilon_d) \end{aligned} \quad (D.2)$$

$$\begin{aligned} h_{AB;CD} &= h_{AB||CD} - \frac{\sqrt{k}}{r}\varepsilon^a(h_{aB||D}\eta_{AC} + h_{Aa||D}\eta_{BC} + h_{aB||C}\eta_{AD} + \\ &\quad h_{Aa||C}\eta_{BD} + h_{AB\ddagger a}\eta_{CD}) + \frac{k}{r^2}(h_{af}\varepsilon^a\varepsilon^f\eta_{AD}\eta_{BC} + h_{af}\varepsilon^a\varepsilon^f\eta_{BD}\eta_{AC} + \\ &\quad -h_{CB}\eta_{AD} - h_{AC}\eta_{BD} - 2h_{AB}\eta_{CD}) \end{aligned} \quad (D.3)$$

$$h_{AB;ab} = h_{AB\ddagger ab} + 2\frac{\sqrt{k}}{r}(h_{AB\ddagger b}\varepsilon_a + h_{AB\ddagger a}\varepsilon_b) + 6\frac{k}{r^2}h_{AB}\varepsilon_b\varepsilon_a - \frac{2m}{r^3}\eta_{ab}h_{AB} \quad (D.4)$$

To simplify the analysis we assume that  $h_{AB} = 0$ . The final gauge-invariant result is not dependent on this assumption but we shall see that formulae are much simpler when we assume this gauge condition. Moreover, we may neglect all terms proportional to the metric  $\eta_{AB}$  because they drop out when we pass to the traceless part. From (D.2–D.4) we obtain

$$\begin{aligned} h^a{}_{B;Aa} &\simeq h^a{}_{B||A\ddagger a} + 2\frac{\sqrt{k}}{r}h_{aB||A}\varepsilon^a \\ h^F{}_{B;AF} &\simeq -4\frac{\sqrt{k}}{r}h_{aB||A}\varepsilon^a \\ h^a{}_{A;Ba} &\simeq h^a{}_{A||B\ddagger a} + 2\frac{\sqrt{k}}{r}h_{aA||B}\varepsilon^a \\ h^F{}_{A;BF} &\simeq -4\frac{\sqrt{k}}{r}h_{aA||B}\varepsilon^a \\ h_{AB;{}^a{}_a} &\simeq 0 \\ h_{AB;{}^F{}_F} &\simeq -\frac{\sqrt{k}}{r}\varepsilon^a(2h_{aB||A} + 2h_{aA||B}) \\ h_{;AB} &\simeq h_{||AB} \end{aligned}$$

where  $\simeq$  denotes an equality modulo trace:

$$t_{AB} \simeq \tau_{AB} \iff TS(t_{AB}) = TS(\tau_{AB}).$$

The above formulae enables one to rewrite (D.1) in a simpler form:

$$2r_{AB} \simeq h^a{}_{B||A\ddagger a} + h^a{}_{A||B\ddagger a} - h_{||AB}. \quad (D.5)$$

One can easily check the following identity on  $S^2$ :

$$\varepsilon^{AC} \left( h_{||AB} - \frac{1}{2}\eta_{AB}h^{||D}{}_D \right)^{||B}{}_C = 0,$$

so the last term in (D.5) drops out when we pass to  $\overset{\circ}{r}_A{}^B{}_{||BC}\varepsilon^{AC}$ . Hence

$$\begin{aligned} 2\overset{\circ}{r}_A{}^B{}_{||BC}\varepsilon^{AC} &= \left( h^a{}_{B||A\ddagger a} + h^a{}_{A||B\ddagger a} - \eta_{AB}\eta^{DE}h^a{}_{D||E\ddagger a} \right) {}^{||B}{}_C\varepsilon^{AC} \\ &= \varepsilon^{AC}\eta^{FB} \left( h^a{}_{B||A\ddagger a} + h^a{}_{A||B\ddagger a} - \eta_{AB}\eta^{DE}h^a{}_{D||E\ddagger a} \right) {}_{||FC}. \end{aligned} \quad (D.6)$$

Taking into account that

$$\eta^{AB}{}_{,a} = 2r(\varepsilon_a\sqrt{k})\eta^{AB}, \quad \varepsilon^{AB}{}_{,a} = 2r(\varepsilon_a\sqrt{k})\varepsilon^{AB}$$

or simply  $(r^2\eta^{AB})_{,a} = (r^2\varepsilon^{AB})_{,a} = 0$ , we get

$$\begin{aligned} 2r^4\overset{\circ}{r}_A{}^B{}_{||BC}\varepsilon^{AC} &= \left[ r^2\varepsilon^{AC}r^2\eta^{FB} \left( h^a{}_{B||A} + h^a{}_{A||B} - \eta_{AB}\eta^{DE}h^a{}_{D||E} \right) {}_{||FC} \right] {}_{\ddagger a} \\ &= \left[ r^2(\overset{\circ}{\Delta} + 2)h^a{}_{A||B}\varepsilon^{AB} \right] {}_{\ddagger a}. \end{aligned} \quad (D.7)$$

Moreover, the gauge condition  $\chi_A{}^B{}_{||BC}\varepsilon^{AC} = 0$  implies that  $\mathbf{y}_a = (\overset{\circ}{\Delta} + 2)h_{aA}{}_{||C}\varepsilon^{AC}$  and we obtain the demanded result (3.39), namely

$$2r^4\overset{\circ}{r}_A{}^B{}_{||BC}\varepsilon^{AC} = (r^2\mathbf{y}^a)_{\ddagger a}. \quad (D.8)$$

The equation (3.38) can be derived in a similar way. Let us start with

$$r_{Bd} = r_{adcB}\eta^{ac} + r_{AdCB}\eta^{AC} = \delta R_{adeB}\eta^{ae} + \delta R_{ABEd}\eta^{AE} - \frac{m}{r^3}h_{dB} \quad (D.9)$$

where the corresponding components of the linearized Riemann tensor  $\delta R_{\mu\nu\lambda\kappa}$  have the following explicit form [10]:

$$\begin{aligned} 2\delta R_{adeB} &= h_{aB\ddagger de} - h_{ae\ddagger d||B} - h_{dB\ddagger ae} + h_{de\ddagger a||B} + \frac{\sqrt{k}}{r}(h_{de||B}\varepsilon_a - h_{ae||B}\varepsilon_d) \\ &\quad + \frac{\sqrt{k}}{r}(h_{aB\ddagger e}\varepsilon_d - h_{dB\ddagger e}\varepsilon_a + h_{Be\ddagger a}\varepsilon_d - h_{Be\ddagger d}\varepsilon_a), \end{aligned} \quad (D.10)$$

$$\begin{aligned} 2\delta R_{ABEd} &= h_{Ad||BE} - h_{Bd||AE} + h_{BE||A\ddagger d} - h_{AE||B\ddagger d} + 2\frac{k}{r^2}\varepsilon^a\varepsilon_d(h_{Aa}\eta_{EB} - h_{Ba}\eta_{EA}) \\ &\quad + \frac{\sqrt{k}}{r}(2h_{BE||A}\varepsilon_d - 2h_{AE||B}\varepsilon_d + h_{ad||A}\varepsilon^a\eta_{BE} - h_{ad||B}\varepsilon^a\eta_{AE}) \\ &\quad + \frac{\sqrt{k}}{r}\varepsilon^a(h_{Bd\ddagger a}\eta_{AE} - h_{Ad\ddagger a}\eta_{BE} + h_{Aa\ddagger d}\eta_{EB} - h_{Ba\ddagger d}\eta_{EA}). \end{aligned} \quad (D.11)$$

Hence from (D.9) we obtain

$$\begin{aligned} 2r_{Bd} &= h^a{}_{B\ddagger da} - h_{dB\ddagger a}{}^a - h_{dB||A}{}^A + 2\frac{\sqrt{k}}{r}(h^a{}_{B\ddagger a}\varepsilon_d - h_{aB\ddagger d}\varepsilon^a) + \frac{k}{r^2}(h_{dB} - 2\varepsilon^a\varepsilon_d h_{aB}) \\ &\quad + \left( h_{BA}{}^{||A} - H_{||B} \right) {}_{\ddagger d} + \left[ h_d{}^a{}_{\ddagger a} - h_d{}^a{}_{\ddagger a} - \frac{\sqrt{k}}{r}h_a{}^a\varepsilon_d + h^A{}_{d||A} \right] {}_{||B}. \end{aligned} \quad (D.12)$$

Let us notice that the all terms in square brackets (gradient) in the above equality do not contribute to axial part  $r_{Bd||D}\varepsilon^{BD}$ . Moreover, if we assume that  $h_{AB} = 0$  (as a gauge condition) we conclude that the “second line” in equation (D.12) drops out and the rest gives

$$2r_{Bd||D}\varepsilon^{BD} = -h_{dB}{}^{||A}{}_{AD}\varepsilon^{BD} + z^a{}_{\ddagger da} - z_{d\ddagger a}{}^a + 4\frac{\sqrt{k}}{r}\varepsilon^a(z_{d\ddagger a} - z_{a\ddagger d}) - \frac{1}{r^2}z_d \quad (D.13)$$



where  $z_a = h_{aA||C}\varepsilon^{AC}$  (cf. B.1)) and we used the following identity:

$$\begin{aligned} & (\varepsilon^{BD}h^a{}_B)_{\dagger da} - (\varepsilon^{BD}h_{dB})^{\dagger a}{}_a + 4\frac{\sqrt{k}}{r}\varepsilon^a[(\varepsilon^{BD}h_{dB})_{\dagger a} - (\varepsilon^{BD}h_{aB})_{\dagger d}] = \\ & = \varepsilon^{BD}\left[h^a{}_{B\dagger da} - h_{dB\dagger a}{}^a + 2\frac{\sqrt{k}}{r}(h^a{}_{B\dagger a}\varepsilon_d - h_{aB\dagger d}\varepsilon^a) + 2\frac{k}{r^2}(h_{dB} - \varepsilon^a\varepsilon_d h_{aB}) + 2\frac{m}{r^3}h_{dB}\right] \end{aligned}$$

implied by (3.25). Using one more identity

$$h_{dB}{}^{||A}{}_{AD}\varepsilon^{BD} = \frac{1}{r^2}(\overset{\circ}{\Delta} + 1)h_{dB||D}\varepsilon^{BD}$$

which is a straightforward consequence of the commutation relation (B.3) for covariant derivatives on  $S^2$ , we get

$$2r_{Bd||D}\varepsilon^{BD} = (z_{a\dagger d} - z_{d\dagger a})^{\dagger a} - \frac{4\sqrt{k}}{r}\varepsilon^a(z_{a\dagger d} - z_{d\dagger a}) - \frac{1}{r^2}(\overset{\circ}{\Delta} + 2)z_d \quad (\text{D.14})$$

or in equivalent form

$$-2r^4 r_{Ba||D}\varepsilon^{BD} = [r^4(z_{a\dagger b} - z_{b\dagger a})]^{\dagger b} + r^2\mathbf{y}_a. \quad (\text{D.15})$$

The identity

$$z_{a\dagger d} - z_{d\dagger a} = \varepsilon_{ad}\varepsilon^{bc}z_{b\dagger c} = \varepsilon_{ad}\frac{\mathbf{y}}{r^2} \quad (\text{D.16})$$

similar to (3.36) implies

$$-2r^4 r_{Bd||D}\varepsilon^{BD} = (r^2\mathbf{y})^{\dagger b}\varepsilon_{ab} + r^2\mathbf{y}_a \quad (\text{D.17})$$

which finally gives equation (3.38) even if we relax the gauge condition  $h_{AB} = 0$  because both sides of (D.17) are gauge independent.

## E Multipoles and traceless tensors

Let  $P^k$  denotes the space of polynomials of degree  $\leq k$  in  $\mathbb{R}^3$ . If function  $f$  is defined in a neighbourhood of a unit sphere  $S^2$ , we can denote by  $Rf$  its restriction to  $S^2$ .

In [6] one can find well-known theorem that  $R(P^k)$  is the direct sum  $\sum_{l=0}^k SH^l$ , where  $SH^l$  denotes the space of spherical harmonics of degree  $l$ , ( $g \in SH^l \iff \overset{\circ}{\Delta}g = -l(l+1)g$ ).

Let  $t$  be a tensor field in a neighbourhood of  $S^2$  in  $\mathbb{R}^3$  and by  $TS(Rt)$  we denote traceless symmetric part of  $Rt$  on  $S^2$ . The following identity holds:

$$TS(R(t_{A_1\dots A_k|B})) = TS((Rt_{A_1\dots A_k})_{||B}) \quad (\text{E.1})$$

where “|” denotes covariant derivative with respect to the flat 3-metric on  $\mathbb{R}^3$  and “||” stands for covariant derivative on  $S^2$ . To prove (E.1) we can observe the following:

$$t_{A_1\dots A_k|B} = t_{A_1\dots A_k||B} + \frac{1}{r}\eta_{A_1B}t_{3A_2\dots A_k} + \frac{1}{r}\eta_{A_2B}t_{A_13\dots A_k} + \dots + \frac{1}{r}\eta_{A_kB}t_{A_1A_2\dots A_{k-1}3}$$

and  $TS(\eta_{AB}t_{A_1\dots A_n}) = 0$ .

We use spherical coordinates in  $\mathbb{R}^3$ :

$$x^3 := r, (x^A), (A = 1, 2), (x^1 = \theta, x^2 = \phi)$$

$$\eta_{AB} = r^2 \overset{\circ}{\gamma}_{AB}, \eta_{33} = 1$$

$$\eta^{AB} = r^{-2} \overset{\circ}{\gamma}^{AB}, \eta^{33} = 1, \eta^{aA} = 0$$

$$\Gamma^3_{AB} = -\frac{1}{r}\eta_{AB}, \Gamma^A_{3B} = \frac{1}{r}\delta^A_B$$

*Theorem:*

$$p \in P^{n-1} \implies TS(Rp|_{A_1 \dots A_n}) = 0$$

*Proof:*

Let us denote by  $(x^k)$  cartesian coordinates in  $\mathbb{R}^3$  such that  $S^2$  corresponds to the surface:  $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ . Transformation rule for transition from spherical to Cartesian coordinates gives as following

$$p|_{A_1 \dots A_n} = x^{i_1}_{,A_1} x^{i_2}_{,A_2} \dots x^{i_n}_{,A_n} p|_{i_1 \dots i_n} = x^{i_1}_{,A_1} x^{i_2}_{,A_2} \dots x^{i_n}_{,A_n} p_{,i_1 \dots i_n},$$

and  $p_{,i_1 \dots i_n} = 0$  because degree of the polynomial  $p$  is not greater than  $n - 1$ . From (E.1) and  $p|_{A_1 \dots A_n} = 0$  we get the result.  $\square$

In particular for  $n = 2$  we can easily obtain that  $\chi^{AB}|_{AB}$  is orthogonal to the space  $SH^0 \oplus SH^1$ . More precisely, if  $f \in SH^0 \oplus SH^1$  then  $TS(f|_{AB}) = 0$  and

$$0 = \int_{S^2} \chi^{AB} TS(f|_{AB}) = \int_{S^2} \chi^{AB}|_{AB} f$$

Let us consider the following diagram:

$$\begin{array}{ccccccccc} V^0 \oplus V^0 & \xrightarrow{i_{01}} & V^1 & \xrightarrow{i_{12}} & V^2 & \xrightarrow{i_{21}} & V^1 & \xrightarrow{i_{10}} & V^0 \oplus V^0 \\ \downarrow Fl & & \downarrow \hat{\cdot} & & \downarrow \hat{\cdot} & & \downarrow \hat{\cdot} & & \downarrow Fl \\ V^0 \oplus V^0 & \xrightarrow{i_{01}} & V^1 & \xrightarrow{i_{12}} & V^2 & \xrightarrow{i_{21}} & V^1 & \xrightarrow{i_{10}} & V^0 \oplus V^0 \end{array}$$

where the mappings and the spaces are defined as follows:

$$i_{01}(f, g) = f|_A + \varepsilon_A^B g|_B$$

$$i_{12}(v) = v_{A||B} + v_{B||A} - \overset{\circ}{\gamma}_{AB} v_{||C}^C$$

$$i_{21}(\chi) = \chi_A^B|_{||B}$$

$$i_{10}(v) = \left( v_{||A}^A, \varepsilon^{AB} v_{A||B} \right)$$

$$Fl(f, g) = (g, f) \quad \hat{v}_A = \varepsilon_A^B v_B \quad \hat{\chi}_{AB} = \varepsilon_A^C \chi_{CB}$$

$V^0$  – scalars on  $S^2$

$V^1$  – covectors on  $S^2$

$V^2$  – symmetric traceless tensors on  $S^2$ .

We have denoted by  $\overset{\circ}{\Delta}$  the laplacian on  $S^2$ . The following equality

$$i_{10} \circ i_{21} \circ i_{12} \circ i_{01} = \overset{\circ}{\Delta} (\overset{\circ}{\Delta} + 2)$$

shows that if we restrict ourselves to the spaces  $\overline{V}^0 := V^0 \ominus [SH^0 \oplus SH^1]$  ( $(\overset{\circ}{\Delta} (\overset{\circ}{\Delta} + 2) \overline{V}^0 = \overline{V}^0$ ) and  $\overline{V}^1 = V^1 \ominus [i_{01}(SH^1)]$  ( $(\overset{\circ}{\Delta} + 1) \overline{V}^1 = \overline{V}^1$ ) then the all mappings in the above diagram become isomorphisms.

We define *mono-dipole-free scalar* as an element of  $\overline{V}^0$ , *mono-dipole-free covector* belongs to  $\overline{V}^1$  and any symmetric traceless tensor on  $S^2$  is *mono-dipole-free*.

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